Supplementary material for the paper "Towards More Efficient and Effective LP-based Algorithms for MRF Optimization"

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Abstract. This supplementary material contains the technical proofs of all theorems of the main paper. It also includes additional experimental results that were omitted from the main paper due to lack of space.

1 Technical proofs

Theorem. *If* $\mathrm{MRF}_{\bar{G}}(\bar{\mathbf{U}}, \bar{\mathbf{P}})$ *is the master MRF resulting from the dual decomposition defined by eqs. (7)-(9), it then holds* $\mathrm{MRF}_{\bar{G}}(\bar{\mathbf{U}}, \bar{\mathbf{P}}) = \mathrm{proj}(\mathrm{MRF}_G(\mathbf{U}, \mathbf{P}))$.

Proof. By comparing Eqs. (9) and (6) in the main paper, it follows directly that the pairwise potentials $\bar{\mathbf{P}}$ and the pairwise potentials of the MRF $\mathrm{proj}(\mathrm{MRF}_G(\mathbf{U}, \mathbf{P}))$ coincide. Therefore, to prove the theorem, it suffices to show that the unary potentials $\bar{\mathbf{U}}$ coincide with the unary potentials of $\mathrm{proj}(\mathrm{MRF}_G(\mathbf{U}, \mathbf{P}))$ as well.

Indeed, due to Eq. (3) it must hold

$$\bar{\mathbf{U}} = \sum_j \bar{\boldsymbol{\theta}}^{\bar{G}_j} \quad .$$

Furthermore, by substituting Eq. (8) into the above equation, it holds that

$$\begin{aligned}
\bar{U}_{\bar{p}}(l) &= \sum\nolimits_j \sum\nolimits_{i:\mathrm{proj}(G_i)=\bar{G}_j} \sum\nolimits_{p:\mathrm{proj}(p)=\bar{p}} \boldsymbol{\theta}_p^{G_i}(l) && (19)\\
&= \sum\nolimits_i \sum\nolimits_{p:\mathrm{proj}(p)=\bar{p}} \boldsymbol{\theta}_p^{G_i}(l) && (20)\\
&= \sum\nolimits_{p:\mathrm{proj}(p)=\bar{p}} \sum\nolimits_i \boldsymbol{\theta}_p^{G_i}(l) && (21)\\
&= \sum\nolimits_{p:\mathrm{proj}(p)=\bar{p}} U_p(l) \quad , && (22)
\end{aligned}$$

where the last equality is due to Eq. (3) that requires $\mathbf{U} = \sum_i \boldsymbol{\theta}^{G_i}$. The theorem now follows directly by comparing Eqs. (22) and (6). □

Theorem. *Let* $\mathrm{Opt}_{\bar{G}}$, Opt_G *denote the optimal values of the dual relaxations for graphs* $\bar{G}$ *and* G *respectively. Then, in general, it holds* $\mathrm{Opt}_{\bar{G}} > \mathrm{Opt}_G$.

Proof. Let OptMRF_G and $\text{OptMRF}_{\bar{G}}$ denote the optimal energies for the two MRFs defined on the graphs G and $\bar{G} = \text{proj}(G)$ respectively. Due to the MRF on $\bar{G}$ being a projection of the MRF on G, it is obvious that, in general, it will hold

$$\text{OptMRF}_{\bar{G}} > \text{OptMRF}_G \ . \tag{23}$$

If we now consider a case where the corresponding dual relaxations are also tight (*e.g.*, both MRFs are submodular), it will then hold

$$\text{Opt}_{\bar{G}} = \text{OptMRF}_{\bar{G}} \tag{24}$$

$$\text{Opt}_G = \text{OptMRF}_G \ . \tag{25}$$

The theorem then follows directly by combining Eqs. (23), (24) and (25). □

Proposition. *If a node p is stabilized then no update of its local dual variables $\{\boldsymbol{\theta}_p^{G_i}(\cdot)\}$ can increase the dual objective. Conversely, if p is non-stabilized, then there always exists an update of the variables $\{\boldsymbol{\theta}_p^{G_i}(\cdot)\}$ that improves the dual.*

Proof. Without loss of generality, we assume that p is contained in only two subgraphs G_1 and G_2. The forward part of the proposition (*i.e.*, the fact that no update of the local dual variables $\{\boldsymbol{\theta}_p^{G_i}(\cdot)\}$ can increase the dual objective when p is a stabilized node) is a direct consequence of the feasibility constraint (3), which requires that the local dual variables at node p satisfy

$$\boldsymbol{\theta}_p^{G_1}(l) + \boldsymbol{\theta}_p^{G_2}(l) = U_p(l) \ , \ \ \forall l \in \mathcal{L} \ . \tag{26}$$

This condition implies that whenever $\boldsymbol{\theta}_p^{G_1}(l)$ is increased by $d\theta$, $\boldsymbol{\theta}_p^{G_2}(l)$ must also decrease by the same amount (and vice versa). Therefore, if $\bar{l}$ is a stable label of p (*i.e.*, a label that is optimal for both slave MRFs on G_1 and G_2), it is then trivial to see that for each increase $d\theta$ of the optimum of the first slave (attained via increasing $\boldsymbol{\theta}_p^{G_1}(\bar{l})$ by $d\theta$), there will also be a decrease of the optimum of the second slave by $d\theta$ (and vice versa). Hence, the total increase of the dual objective will be zero in the best case.

To prove the converse statement, we will show that if p is non-stabilized then we can increase the dual objective by properly updating the local dual variables at p. Indeed, since p is non-stabilized, there will exist no label l that is optimal for both of the slave MRFs on G_1 and G_2. If we then denote by $\bar{\mathcal{L}}_1$ the set of optimal labels for p with regard to the slave MRF on G_1, it is easy to show that the following feasible update of the local dual variables increases the dual objective by $d\theta > 0$ for a small enough $d\theta$:

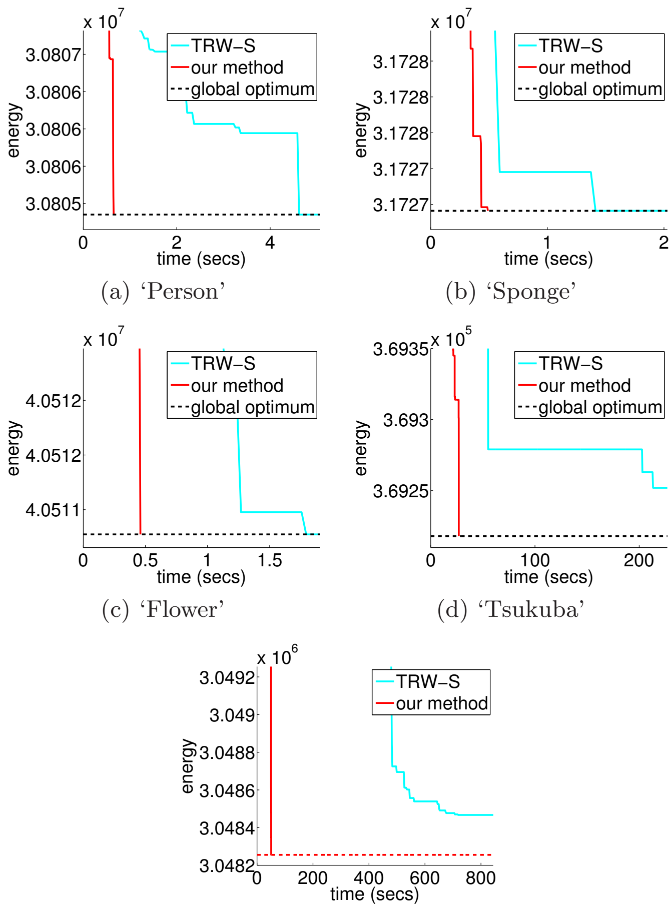
$$\boldsymbol{\theta}_p^{G_1}(l) = \boldsymbol{\theta}_p^{G_1}(l) + d\theta, \quad \forall l \in \bar{\mathcal{L}}_1 \tag{27}$$

$$\boldsymbol{\theta}_p^{G_2}(l) = \boldsymbol{\theta}_p^{G_2}(l) - d\theta, \quad \forall l \in \bar{\mathcal{L}}_1 \ . \tag{28}$$

More specifically, for a small enough $d\theta$, the above update increases the optimum of the first slave by $d\theta$, while it leaves unmodified the optimum of the second slave. □

2 Additional experiments

We also compared our method with the original implementation of the TRW-S algorithm by V. Kolmogorov (available from `http://www.cs.ucl.ac.uk/staff/V.Kolmogorov/papers/TRW-S.html`). To this end, we provide below the resulting “energy vs time” plots for various problems from the “Middlebury” benchmark dataset.



(a) ‘Person’ (b) ‘Sponge’ (c) ‘Flower’ (d) ‘Tsukuba’ (e) ‘Venus’

Fig. 1: “Energy vs time” plots for various benchmark problems from “Middlebury”