PLANAR GRAPHS

Basic definitions

Isomorphic graphs
Two graphs $G1(V1,E1)$ and $G2(V2,E2)$ are *isomorphic* if there is a one-to-one correspondence F of their vertices such that the following holds:
- $\forall u,v \in V1$, $uv \in E1$, $=> F(u)F(v) \in E2$
- $\forall x,y \in V1$, $xy \notin E1$ $=> F(x)F(y) \notin E2$

Plane graph (or embedded graph)
A graph that is drawn on the plane without edge crossing, is called a *Plane graph*

Planar graph
A graph is called *Planar,* if it is isomorphic with a *Plane graph*

Phases
A planar representation of a graph divides the plane in to a number of connected regions, called *faces,* each bounded by edges of the graph.

For every graph G, we denote $n(G)$ the number of vertices , $e(G)$ the number of edges, $f(G)$ the number of faces.

Degree
We define the *degree of a face d(f)*, to be the number of edges bounding the face f.

Examples

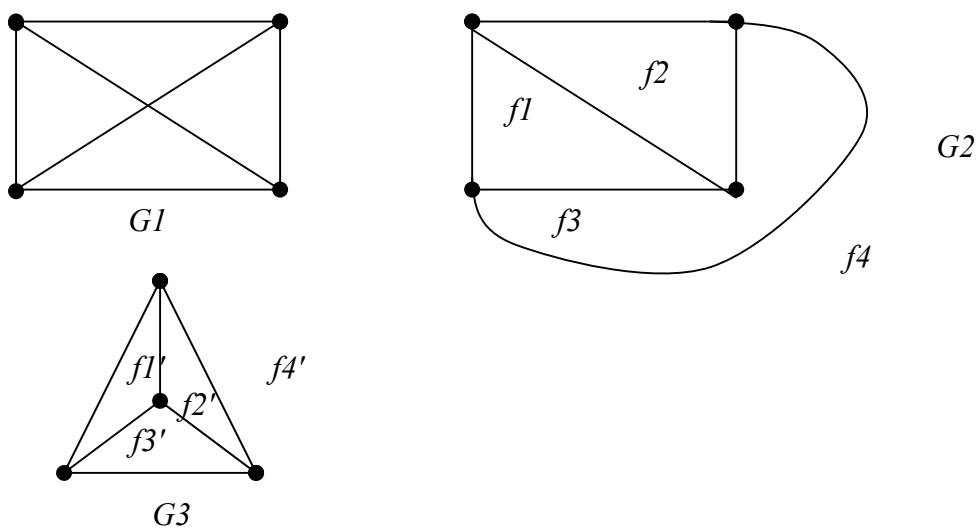The following graphs are isomorphic to 4 (the *complete* graph with 4 vertices)

Figure 1: 3 isomorphic graphs

G2 and *G3* are plane graphs, *G1* is not.
G1 is planar, as it is isomorphic to *G2 and G3*.
G2 and G3 are called *planar embeddings* of *K*4

f1, f2 and f3 (exterior phase) are phases of G2, with degree *d(f1)=d(f2)=d(f3)=3*.

Theorem 1   *A graph is embeddable in the sphere if and only if it is embeddable in the plane.*

Proof. We show this by using a mapping known as *stereographic projection.* Consider a spherical surface *S* touching a plane *P* at the point *SP* (called *south pole). The point *NP* (called the *point of projection* or *north pole)* is on *S* and diametrically opposite *SP*. Any point *z* on *P* can be projected uniquely onto *S* at *z′* by making *NP*, *z* and *z′* collinear. In this way any graph embedded in *P* can be projected onto *S*. Conversely, we can project any graph embedded in *s* onto *P* , choosing *NP* so as not to lie an any vertex or edge of the graph.

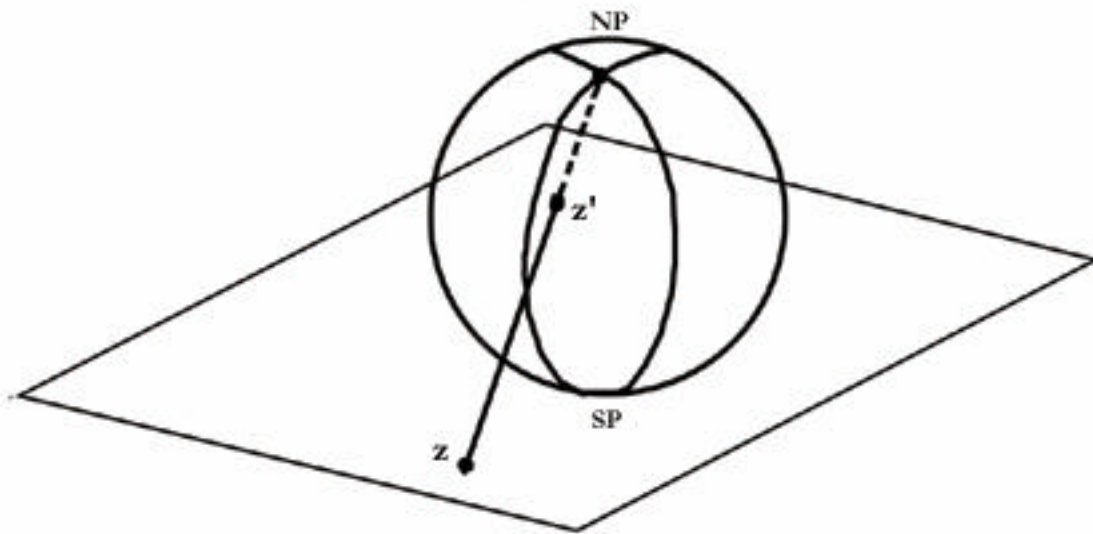

figure 2: graph embeddable in sphere

Theorem 2   *A planar embedding G' of a graph G can be transformed in to another embedding such that any specified face becomes the exterior face.*

Proof. Any face of *G'* is defined by the path which forms its boundary. Any such path, T, identified in a particular planar representation *P* of *G,* may be made to define the exterior face of a different planar representation *P'* as follows. We form a spherical embedding *P''* of *P. P'* is then formed by projecting *P''* onto the plane in such a way that the point of projection lies in the face defined by the image of *T* on the sphere.

Example

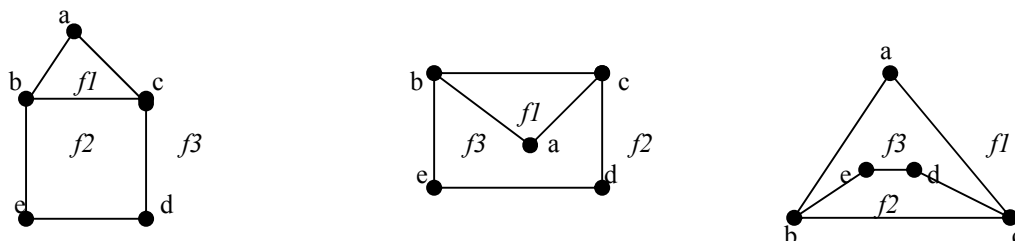

figure 3

Theorem 3 (Euler's formula) *If G is a connected planar graph, for any embedding G'*
the following formula holds: n(G)+f(G) = e(G)+2

Proof. By induction on *f*.
- For *f(G) = 1, G* is a tree. For every tree, *e(G) = n(G)-1,* so *n(G)+1= e(G) + 2* $\Rightarrow$ *n(G)+f(G) =e(G) +2* and the formula holds.
- Suppose it holds for all planar graphs with less than *f* faces and suppose that *G'* has *f ≥2* faces.
- Let *(u,v)* be an edge of *G* which is not a cut-edge. Such an edge must exists because *G'* has more than one face. The removal of *(u.v)* will cause the two faces separated by *(u,v)* to combine, forming a single face.
  Hence *(G-(u,v))'* is a planar embedding of a connected graph with one less face than *G'* , hence:

  f(G - (u,v)) = f(G) - 1
  n(G - (u,v) = n(G)
  e(G = (u,v)) = e(G) - 1

  But by the induction hypothesis:
  n(G- (u,v)) + f(G - (u,v)) = e(G - (u,v))   + 2
  and so, by substitution:
  n(G) + f(G) = e(G) + 2

Hence, by induction, Euler's formula holds for all connected planar graphs.

Lemma 1 *For any embedding G' of any simple connected planar graph G,*
$$\sum_i d(f_i) = 2e(G)$$

Proof. Each edge contributes 1 to each face it is a bound, so it contributes 2 to the total sum. So the *e(G)* edges contributes 2e(*G)* to the total sum.

Lemma 2 *For any simple connected planar graph G, with e(G) ≥3, the following holds:*
e(G) ≤ 3n(G) - 6

Proof. Each face of any embedding *G'* of *G* is bounded by at least three edges, hence:
$$\sum_i d(f_i) \geq 3f(G)$$

Form the above lemma, $\sum_i d(f_i) = 2e(G)$, hence:

2e(G) ≥3f(G) => f(G) ≤ $\frac{2}{3}$ e(G)
From Euler's formula, *n(G)+f(G) = e(G)+2 ,* so
n(G) + $\frac{2}{3}$ e(G) ≥ e(G) + 2 $\Rightarrow$ $\frac{1}{3}$ e(G) ≤ n(G) - 2 $\Rightarrow$ e(G) ≤3n(G) - 6

Definitions

Bipartite graph
A bipartite graph is a graph with no cycles of odd number of edges.
In a bipartite graph, the set of vertices can be partitioned to two disjoint not empty
subsets V1 and V2, so that every edge of V1 connects a vertex of V1 with a vertex of V2.
Complete bipartite graph
A *complete bipartite graph, denoted as Km,n* is a bipartite graph where V1 has m vertices, V2
has n vertices and every vertex of each subset is connected with all other vertices of the other
subset. Km,n haw m+n vertices and m*n edges.

Corollary 1 *A simple connected planar bipartite graph, has each face with even degree.*

Proof. Each face is a cycle and the graph is bipartite, so each face
 must has even number of vertices.

Lemma 3 *For any simple connected bipartite graph G, with e(G) ≥ 3. the following holds:*
e(G) ≤ 2n(G) - 4

Proof. G is bipartite, so each face of every embedding G′ has at least 4 edges, hence
$$\sum_i d(f_i) \geq 4 f(G)$$

For every simple connected planar graph, $\sum_i d(f_i) = 2e(G)$, hence:

$2e(G) \geq 4f(G) \Rightarrow f(G) \leq \frac{1}{2} \ e(G)$
From Euler's formula, $n(G)+f(G) = e(G)+2$, hence
$n(G) + \frac{1}{2} \ e(G) \geq e(G) + 2 \Rightarrow \frac{1}{2} \ e(G) \leq n(G) - 2 \Rightarrow e(G) \leq 2n(G) - 4$

Examples

K5

a) *K5* is not planar.

n(K5) = 5

$e(K5) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$

3n(K5) - 6 = 15 - 6 = 9.

figure 4

e(K5) > 3n(K5) - 6, hence *K5* is not planar.
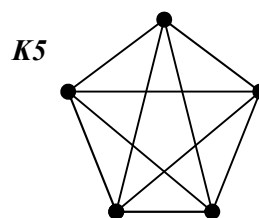
b) K3,3 is not planar.   **K3,3**

n(K3,3) = 6
e(K3,3) = 3*3=9
2n(K3,3) - 4 = 12-4 = 8
e(K3,3) > 2n(K3,3) - 4, hence K3,3 is not planar.

c)   figure 5
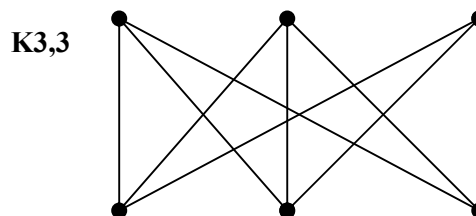The formula *e(G) ≤ 3n(G) - 6* does not holds for *e(G) < 3.*
K1 has n(K1)=1, e(K1)=0 and e(K1) >3n(K1) - 6
K2 has n(K2)=2, e(K2)=1 and e(K2) > 3n(K2) - 6

Definitions

Genus
For any non-negative integer g, we can construct a surface in which it is possible to embed g
non-intersecting closed curves without separating the surface into two regions. If for the same
surface (g+1) closed curves always cause a separation, then the surface is said to have a *genus*
equal to g.
A graph that can be embedded in a surface of genus g , but not on a surface of genus (g-1) is
called *a graph of genus g.*

Crossing number
The *Crossing number* of a graph is the minimum number of crossings of edges for the graph
drawn in the plane.

Thickness

The thickness $T(G)$ of a graph G is the minimum number of planar sub-graphs of G whose union is G.

Corollary 2

The thickness T of a simple graph G satisfies

$$T(G) \geq \frac{e(G)}{3n(G)-6}$$

Proof. Each planar sub-graph will contain at most 3n*(G)-1* edges.
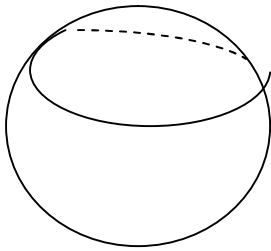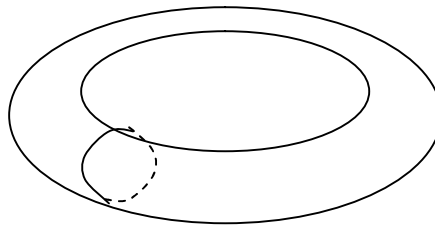
Examples

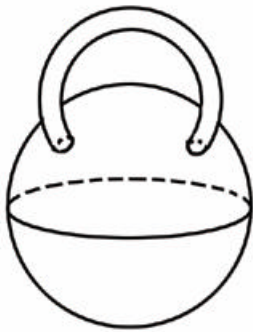figure 6: a sphere of *genus* 0

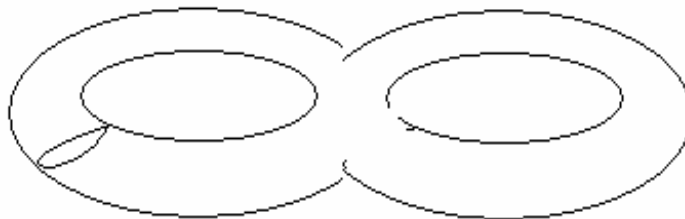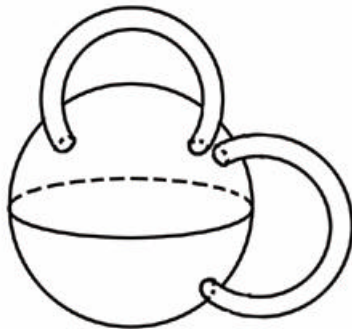figure 7: a sphere with a 'handle'   figure 8: a torus, are both *genus* 1.

figure 9: a sphere with two 'handles' figure 10: a double torus, are both *genus 2*
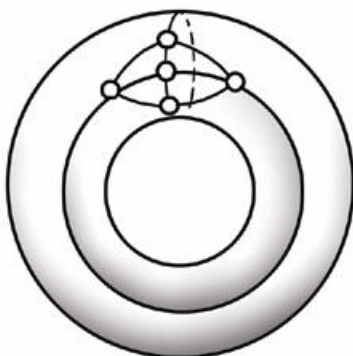
figure 11: a graph of genus 1.

Figure 12: *K5* has crossing number 1.
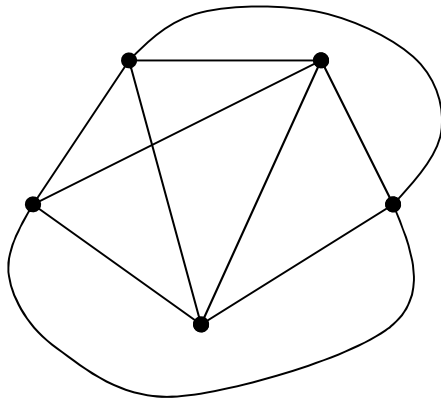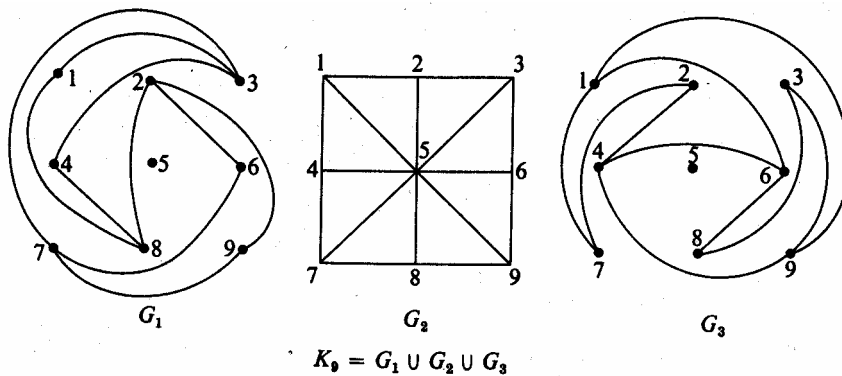

$$K_9 = G_1 \cup G_2 \cup G_3$$

figure 13:   K9 is the union of three planar graphs, hence T(K9) ≤ 3

Theorem 4 *If G is a connected graph with genus g, then: f(G) = e(G) – n(G) + 2 –2g*

Proof. By induction on g.
- For g=0 the graph is planar and we have *f(G) = e(G) – n(G) + 2 – 2g*
  from Euler's formula.
- We assume that the theorem is true for all graphs with genus (g-1)
  These graphs may be drawn on a spherical surface with (g-1) handles and include
  all those graphs obtained by deleting those edges passing over a single handle in
  any graph of genus g.
- We construct *G* with genus g on a surface o genus g by adding a single edge to *G'*,
  requiring an additional handle. Using prime letters for G', we have by the induction
  hypothesis:
  $F(G') = e(G') – n(G') + 2 – 2g'$
  But $e(G)=e(G')+1$, $g = g'+1$, $n-n(G') +1$
  Also $f(G) = f(G') – 1$ because the handle connects two distinct faces in *G'* making
  A single face in *G*. Hence by substitution:
  $F(G) = e(G) – n(G) + 2 – 2 g.$
  And so by induction the theorem is proved.

Corollary 3

If *G* is a connected graph with genus *n(G) ≥ 4* then
$g \geq \frac{1}{6} (e(G) – 3n(G)) + 1$
Proof.
From the above theorem we get:
$g= \frac{1}{2} (e(G) – n(G) - f(G)) + 1$

Every face of an embedding of the graph is bound by at least three edges of which separates two faces, therefore $3f(G) \leq 2e(G)$, and so the result follows by substitution.

Definition

Homeomorphic graphs
Two graphs are said to be *homeomorphic* if one can be made isomorphic to the other by the addition or the deletion of vertices of degree two, in the following manner:
By dividing an edge into two edges in series by the insertion of a vertex of degree 2, or by reverse of this process;.
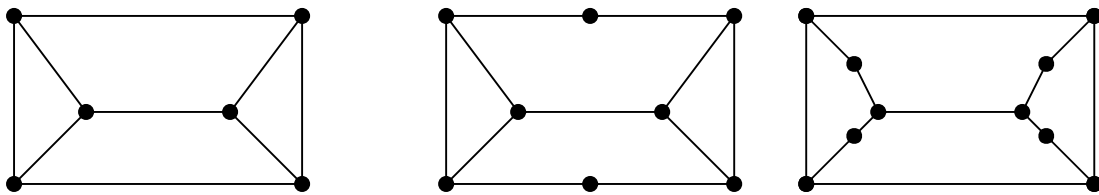
Example

Figure 14: homeomorphic graphs

Theorem 5 (Kuratowski) *A graph is planar if and only if it has no sub-graph homeomorphic to K5 or to K3,3.*

Proof. The proof is out of the subject of this course. For a proof you can look at Alan Gibbons book, "Algorithmic graph theory", page 77.
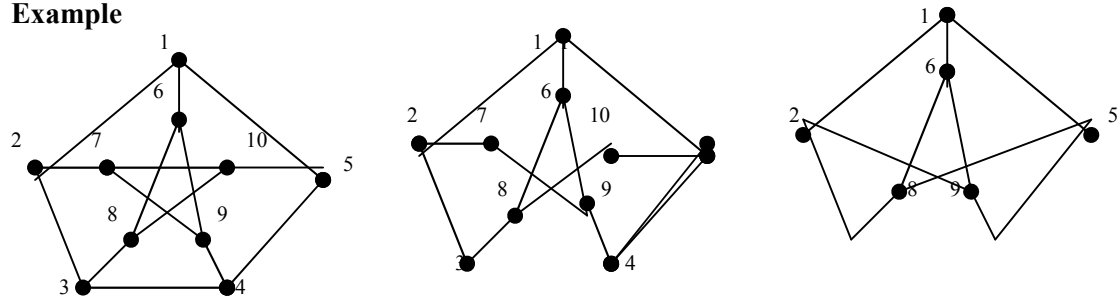
Example

figure 15: graphs G1, G2 and G3

Graph G1 is not planar, since it has a sub-graph (*G2)* homeomorphic to *G3,*
Which is isomorphic to K3,3 (The partition of *G3* vertices is{ 1,8,9} and {2,5,6})

Definitions

Coloring
A *coloring* of the vertices of a graph is a mapping of any vertex of the graph to a color such that any vertices connected with an edge have different colors.
The minimum number of colors required for a graph coloring is called *coloring number* of the graph.

Theorem 6 (Four Color theorem)
A planar graph has coloring number ≤ 4

The proof was given in 1976 by Appel and Haken, using computers and analyzing the problem to 1936 different cases.

Lemma 4 *The four color theorem is equivalent to the following lemma:*
The vertices of a planar graph can be partitioned in two sets, V1,V2 such that G/V1,
G/V2 are both bipartite graphs.

Proof.
 a) 4 Color Theorem => Lemma
I can get immediately the two bipartite graphs by choosing 2 of the four colors for one of the
bipartite graphs.
 b) Lemma => 4 Color theorem
 I can get the two bipartite graphs and put 2 different colors on each graph.

Definition

Dual graph.
Let G be a planar graph and a planar embedding G'
A dual graph G* has a vertex for each face of G' and an edge between two faces $f1$ and $f2$ if
and only if $f1$ and $f2$ share an edge.

Dual graph is not always a simple graph.
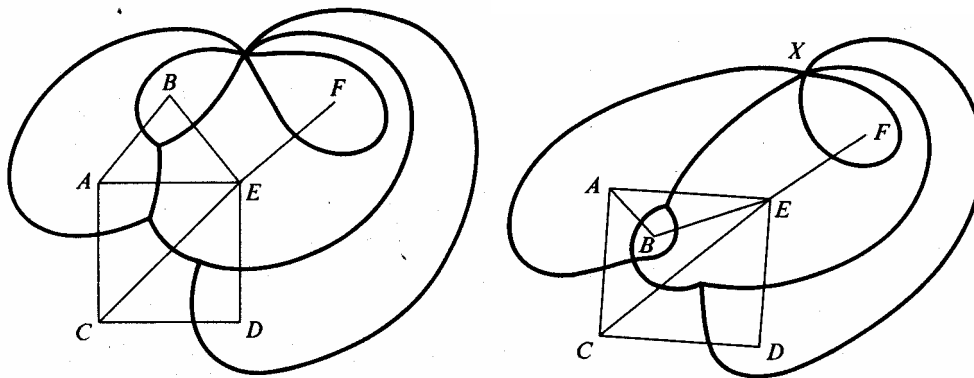
Example

figure 16

 We can see that the dual of the first planar representation of the graph, is not isomorphic to
 the dual of the second representation.
 The dual graph refers to of a planar representation and not to the graph.

Lemma 5 *A graph is planar if and only if the dual graphs of it's planar embeddings are*
planar

For a proof you can look at Alan Gibbons book, "Algorithmic graph theory", page 83.

Corollary 4 *Using dual graphs, w can see that the four color theorem is equivalent to: We*
can color the faces of a graph such that two faces who share an edge have different color,
using at most four colors.

Planarity Testing

A sophisticated algorithm given by Hopcroft and Tarjan, tests planarity in *O(n)* time.
Here we will see a simple algorithm that test planarity $O(n^3)$ time.

As a first application of the divide and conquer principle , we observe that:

- A graph is planar if and only if all its connected components are planar
- A connected graph is planar if and only if all its biconnected components are planar

Thus, via preliminary decomposition into connected and biconnected components, we can restrict our attention to the problem of testing the planarity of a biconnected graph.

Definitions

Piece
Let G be a biconnected graph.
Let C be a cycle in G.
A *piece* is:
   a) an edge with vertices on C
   b) a connected component after removing the edges of C

Separating cycle
A cycle C of G is said to be *separating* if it has at least two peaces, and is called *nonseparating* if it has one peace. Of course, if $G = C$, then C has no pieces.

In any biconnected graph there is a separating cycle and from any non-separating cycle, I can always get a separating cycle.
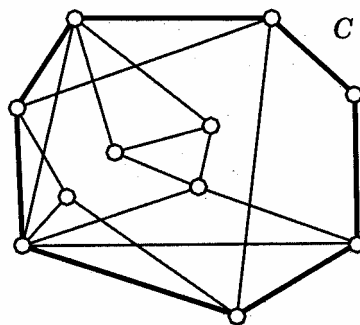
Example
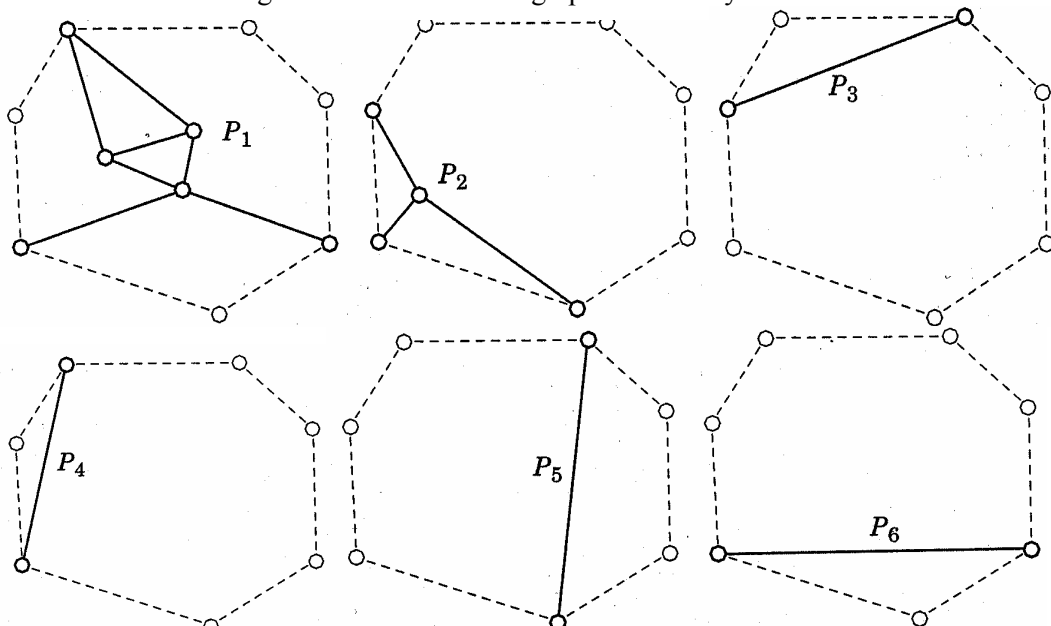

figure 17: a biconnected graph G and a cycle C.

figure 18: the pieces of G with respect *to C*

Lemma 5 *Let G be a biconnected graph and let C be a nonseparating cycle of G with piece P.\If P is not a path, then G has a separating cycle C' consisting of a subpath of C plus a path of P between two attachments.*

Proof. Let *u* and *v* be two attachments of *P* that are consecutive in the circular ordering, and let g be a subpath of *C* between *u* and *v* that does not contain any attachment of *C*. Since *P* is connected, there is a path *p*, between *u* and v. Let *C'* be the cycle obtained from *C* by replacing *g* with *p*. We have that *g* is a piece of *G* with respect to *C'*. If *P* is not a path, let *e* be an edge of *P* not in *p*. There is a pieces of *C'* distinct from *g* containing *e*. Thus, if *P* is not a path, then *C'* has at least two pieces and is thus a separating cycle of *G*.
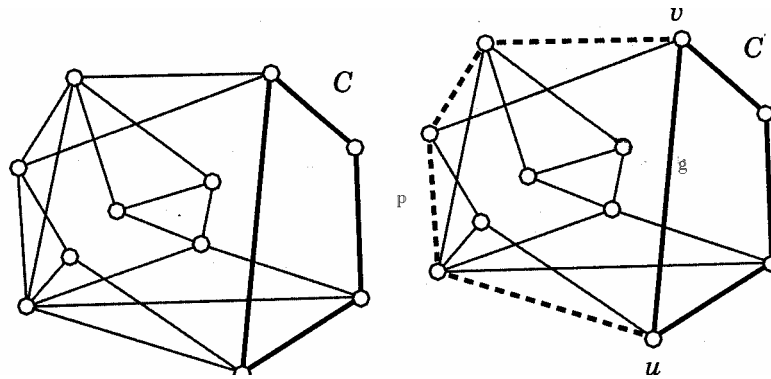
Example

figure 19: a nonseparating cycle C and a separating cycle C' obtained from C as shown in the proof of the Lemma.

Definition

The interlacement graph

If the graph *G* is planar, then in any planar drawing of *G* each peace is drawn either inside *C* or entirely outside *c*. We say that two pieces of *G*, with respect to *C, interlace* if they cannot be drawn on the same side of *C* without violating planarity.

The *interlacement graph* of the pieces of *G,* with respect to *C*, is the graph whose vertices are the pieces of *G* and whose edges are the pairs of pieces that interlace.
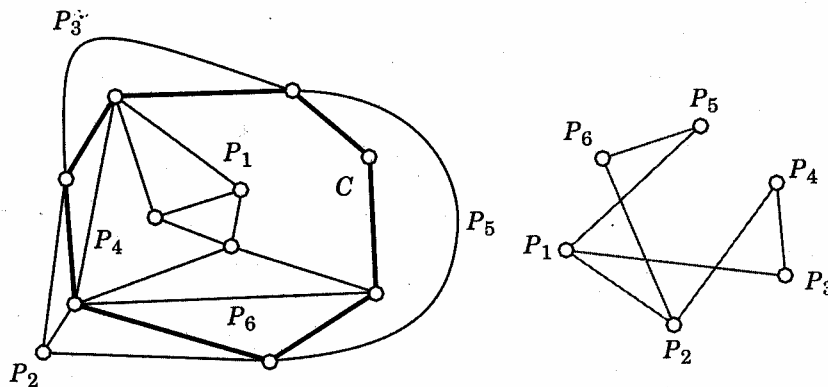
Example

figure 20

A planar drawing of the above graph, where pieces *P1, P3 and P6* are drawn inside cycle *C* and the other pieces are drawn outside and the interlacement graph *I* of the pieces of *F* with respect to cycle *C*

Theorem 7 *A biconnected graph G with a cycle C is planar if and only if the following conditions hold:*

- For each piece P of G with respect to C, the graph obtained by adding P to C is planar
- The interlacement graph of the pieces of G, with respect to C, is bipartite.

Algorithm

The correctness of the algorithm is based on the above lemma and theorem and on the fact that the graph P' obtained by adding a piece P to C is biconnected.

Input: a biconnected graph, G, with n vertices and at most 3n-6 edges, and a separating cycle C of G.
Output: and indication of whether G is planar.

1. **Compute** the pieces of G with respect to C.
2. **for each** piece P of G that is not a path (of one or more edges):
   (a) **let** P' be the graph obtained by adding P to C
   (b) **let** C' be the cycle of P' obtained from C by replacing the portion of C between two consecutive attachments with a path of P between them
   (c) **apply** the algorithm recursively to graph P' and cycle C'. **If** P' is non planar, **return** "nonplanar"
3. **Compute** the interlacement graph I of the pieces.
4. **Test** whether I is bipartite. **If** is not bipartite, **return** "nonplanar"
5. **Return** "planar"

Time analysis

Step 1: $O(n)$
Step 2: $O(n))$
Step 3: $O(n^2)$
Step 4: $O(n^2)$

So a recursive invocation of algorithm of planarity testing takes $O(n^2)$ time.
We observe that number of recursive invocations as $O(n)$ by associating with each invocation a distinct edge of G. . We conclude that the running time of algorithm Planarity Testing is $O(n^3)$

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