

PLANAR GRAPHS

(lecture notes for HY483, by Fountouris Antonios and Akkus Zebide)

Basic definitions

Isomorphic graphs

Two graphs $G1(V1,E1)$ and $G2(V2,E2)$ are *isomorphic* if there is a one-to-one correspondence F of their vertices such that the following holds:

- $\forall u,v \in V1, uv \in E1, \Rightarrow F(u)F(v) \in E2$
- $\forall x,y \in V1, xy \notin E1 \Rightarrow F(x)F(y) \notin E2$

Plane graph (or embedded graph)

A graph that is drawn on the plane without edge crossing, is called a *Plane graph*

Planar graph

A graph is called *Planar*, if it is isomorphic with a *Plane graph*

Phases

A planar representation of a graph divides the plane into a number of connected regions, called *faces*, each bounded by edges of the graph.

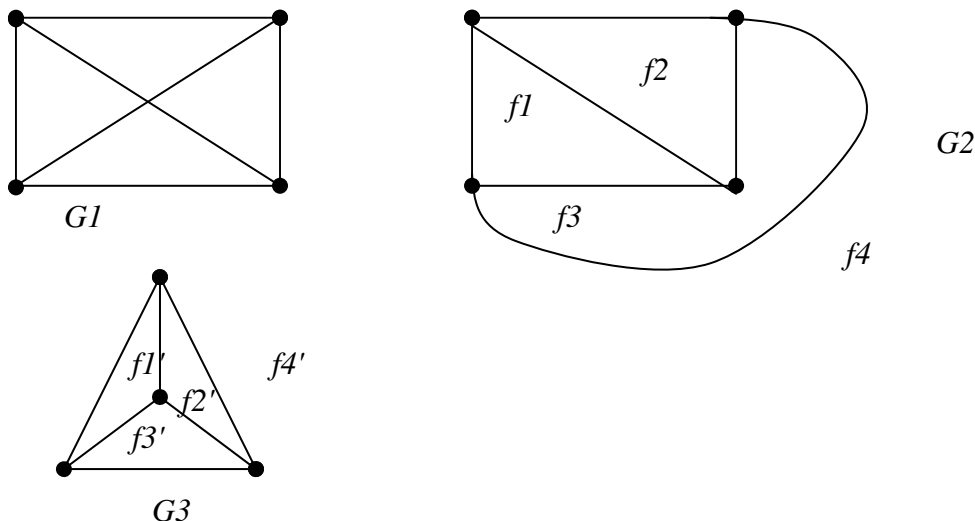
For every graph G , we denote $n(G)$ the number of vertices, $e(G)$ the number of edges, $f(G)$ the number of faces.

Degree

We define the *degree of a face* $d(f)$, to be the number of edges bounding the face f .

Examples

The following graphs are isomorphic to $K4$ (the *complete* graph with 4 vertices)



$G2$ and $G3$ are plane graphs, $G1$ is not.

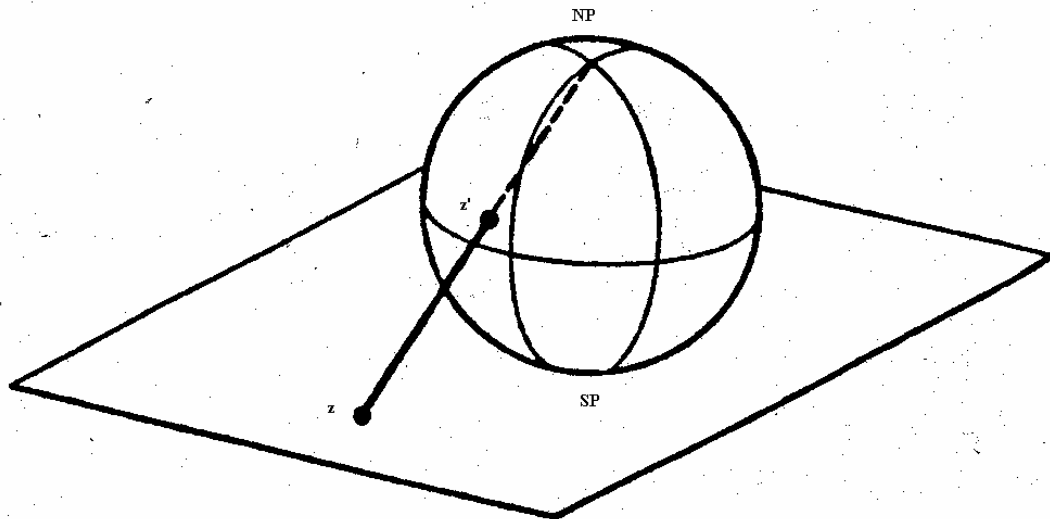
$G1$ is planar, as it is isomorphic to $G2$ and $G3$.

$G2$ and $G3$ are called *planar embeddings* of $K4$

$f1, f2$ and $f3$ (exterior phase) are phases of $G2$, with degree $d(f1)=d(f2)=d(f3)=3$.

Theorem

A graph is embeddable in the sphere if and only if it is embeddable in the plane.



Proof. We show this by using a mapping known as *stereographic projection*.

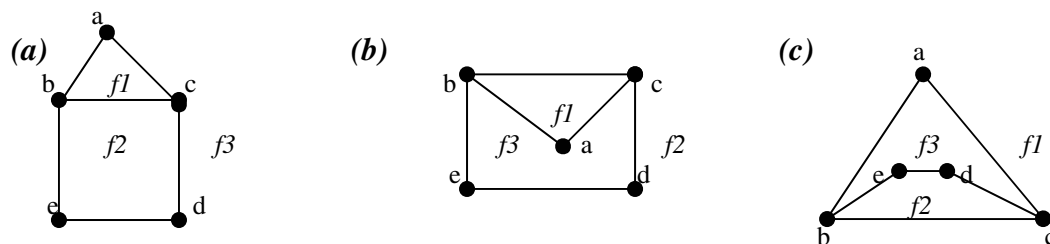
Consider a spherical surface S touching a plane P at the point SP (called *south pole*). The point NP (called the *point of projection* or *north pole*) is on S and diametrically opposite SP . Any point z on P can be projected uniquely onto S at z' by making NP, z and z' collinear. In this way any graph embedded in P can be projected onto S . Conversely, we can project any graph embedded in s onto P , choosing NP so as not to lie on any vertex or edge of the graph.

Theorem

A planar embedding G' of a graph G can be transformed into another embedding such that any specified face becomes the exterior face.

Proof. Any face of G' is defined by the path which forms its boundary. Any such path, T , identified in a particular planar representation P of G , may be made to define the exterior face of a different planar representation P' as follows. We form a spherical embedding P'' of P . P' is then formed by projecting P'' onto the plane in such a way that the point of projection lies in the face defined by the image of T on the sphere.

Example



Theorem (Euler's formula)

If G is a connected planar graph, for any embedding G' the following formula holds:

$$n(G) + f(G) = e(G) + 2$$

Proof. By induction on f .

- For $f(G) = 1$, G is a tree. For every tree, $e(G) = n(G) - 1$, so $n(G) + 1 = e(G) + 2 \Rightarrow n(G) + f(G) = e(G) + 2$ and the formula holds.
- Suppose it holds for all planar graphs with less than f faces and suppose that G' has $f \geq 2$ faces.
- Let (u, v) be an edge of G which is not a cut-edge. Such an edge must exist because G' has more than one face. The removal of (u, v) will cause the two faces separated by (u, v) to combine, forming a single face. Hence $(G - (u, v))'$ is a planar embedding of a connected graph with one less face than G' , hence:

$$\begin{aligned} f(G - (u, v)) &= f(G) - 1 \\ n(G - (u, v)) &= n(G) \\ e(G - (u, v)) &= e(G) - 1 \end{aligned}$$

But by the induction hypothesis:

$$n(G - (u, v)) + f(G - (u, v)) = e(G - (u, v)) + 2$$

and so, by substitution:

$$n(G) + f(G) = e(G) + 2$$

Hence, by induction, Euler's formula holds for all connected planar graphs.

Lemma

For any embedding G' of any simple connected planar graph G ,

$$\sum_i d(f_i) = 2e(G)$$

Proof. Each edge contributes 1 to each face it is a bound, so it contributes 2 to the total sum. So the $e(G)$ edges contribute $2e(G)$ to the total sum.

Lemma

For any simple connected planar graph G , with $e(G) \geq 3$, the following holds:

$$e(G) \leq 3n(G) - 6$$

Proof. Each face of any embedding G' of G is bounded by at least three edges, hence:

$$\sum_i d(f_i) \geq 3f(G)$$

From the above lemma, $\sum_i d(f_i) = 2e(G)$, hence:

$$2e(G) \geq 3f(G) \Rightarrow f(G) \leq \frac{2}{3} e(G)$$

From Euler's formula, $n(G) + f(G) = e(G) + 2$, so

$$n(G) + \frac{2}{3} e(G) \geq e(G) + 2 \Rightarrow \frac{1}{3} e(G) \leq n(G) - 2 \Rightarrow e(G) \leq 3n(G) - 6$$

Definitions

Bipartite graph

A bipartite graph is a graph with no cycles of odd number of edges.

In a bipartite graph, the set of vertices can be partitioned to two disjoint not empty subsets V_1 and V_2 , so that every edge of V_1 connects a vertex of V_1 with a vertex of V_2 .

Complete bipartite graph

A complete bipartite graph, denoted as $K_{m,n}$ is a bipartite graph where V_1 has m vertices, V_2 has n vertices and every vertex of each subset is connected with all other vertices of the other subset. $K_{m,n}$ has $m+n$ vertices and $m*n$ edges.

Corollary

A simple connected planar bipartite graph, has each face with even degree.

Proof. Each face is a cycle and the graph is bipartite, so each face must have even number of vertices.

Lemma

For any simple connected bipartite graph G , with $e(G) \geq 3$, the following holds:

$$e(G) \leq 2n(G) - 4$$

Proof. G is bipartite, so each face of every embedding G' has at least 4 edges, hence

$$\sum_i d(f_i) \geq 4f(G)$$

For every simple connected planar graph, $\sum_i d(f_i) = 2e(G)$, hence:

$$2e(G) \geq 4f(G) \Rightarrow f(G) \leq \frac{1}{2} e(G)$$

From Euler's formula, $n(G)+f(G) = e(G)+2$, hence

$$n(G) + \frac{1}{2} e(G) \geq e(G) + 2 \Rightarrow \frac{1}{2} e(G) \leq n(G) - 2 \Rightarrow e(G) \leq 2n(G) - 4$$

Examples

a) K_5 is not planar.

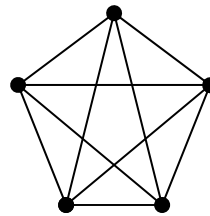
$$n(K_5) = 5$$

$$e(K_5) = \binom{5}{2} = 10$$

$$3n(K_5) - 6 = 15 - 6 = 9.$$

$e(K_5) > 3n(K_5) - 6$, hence K_5 is not planar.

K_5



b) $K_{3,3}$ is not planar.

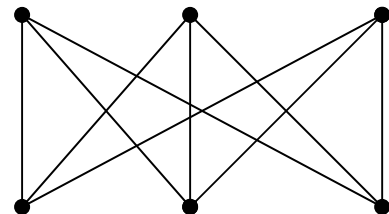
$$n(K_{3,3}) = 6$$

$$e(K_{3,3}) = 3*3=9$$

$$2n(K_{3,3}) - 4 = 12-4 = 8$$

$e(K_{3,3}) > 2n(K_{3,3}) - 4$, hence $K_{3,3}$ is not planar.

$K_{3,3}$



c)

The formula $e(G) \leq 3n(G) - 6$ does not hold for $e(G) < 3$.

K_1 has $n(K_1)=1$, $e(K_1)=0$ and $e(K_1) > 3n(K_1) - 6$

K_2 has $n(K_2)=2$, $e(K_2)=1$ and $e(K_2) > 3n(K_2) - 6$

Definitions

Genus

For any non-negative integer g , we can construct a surface in which it is possible to embed g non-intersecting closed curves without separating the surface into two regions. If for the same surface $(g+1)$ closed curves always cause a separation, then the surface is said to have a *genus* equal to g .

A graph that can be embedded in a surface of genus g , but not on a surface of genus $(g-1)$ is called a *graph of genus* g .

Crossing number

The *Crossing number* of a graph is the minimum number of crossings of edges for the graph drawn in the plane.

Thickness

The thickness $T(G)$ of a graph G is the minimum number of planar sub-graphs of G whose union is G .

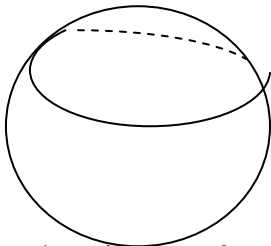
Corollary

The thickness T of a simple graph G satisfies

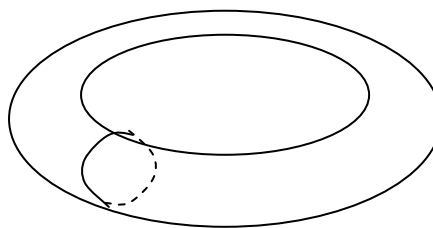
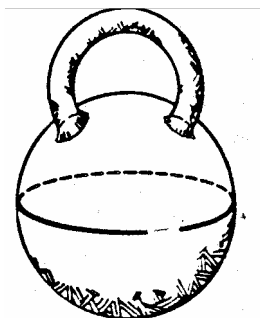
$$T(G) \geq \frac{e(G)}{3n(G)-6}$$

Proof. Each planar sub-graph will contain at most $3n(G)-6$ edges.

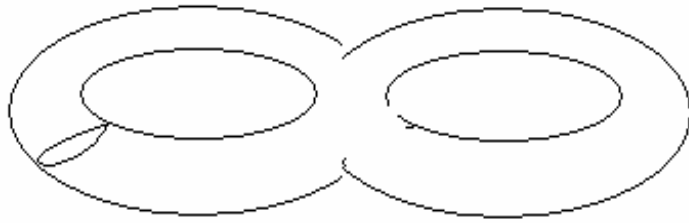
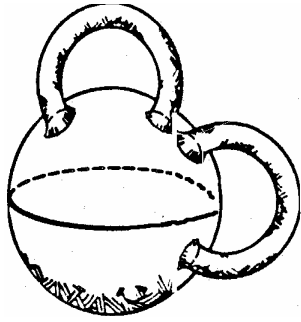
Examples



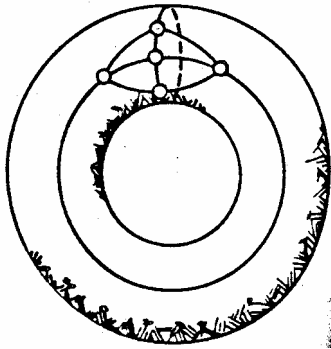
A sphere is *genus* 0



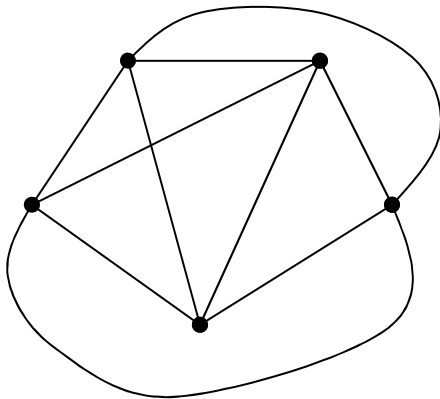
A sphere with a 'handle' and a torus, are both *genus* 1.



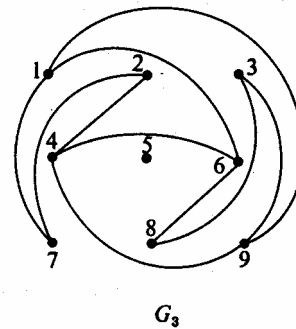
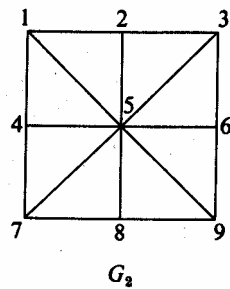
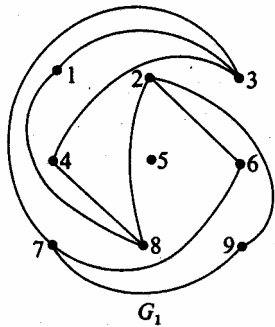
A sphere with two 'handles' and a double torus, are both *genus 2*



K_5 can be embedded in torus, but not to the sphere or the plane, it's a graph of genus 1.



K_5 can be drawn in the plane with minimum number of 1 edge crossing, it has crossing number 1.



$$K_9 = G_1 \cup G_2 \cup G_3$$

K_9 is the union of three planar graphs, hence $T(K_9) \leq 3$

Theorem

If G is a connected graph with genus g , then:

$$f(G) = e(G) - n(G) + 2 - 2g$$

Proof. By induction on g .

- For $g=0$ the graph is planar and we have $f(G) = e(G) - n(G) + 2 - 2g$ from Euler's formula.
- We assume that the theorem is true for all graphs with genus $(g-1)$. These graphs may be drawn on a spherical surface with $(g-1)$ handles and include all those graphs obtained by deleting those edges passing over a single handle in any graph of genus g .
- We construct G with genus g on a surface of genus g by adding a single edge to G' , requiring an additional handle. Using prime letters for G' , we have by the induction hypothesis:

$$F(G') = e(G') - n(G') + 2 - 2g'$$

$$\text{But } e(G) = e(G') + 1, \quad g = g' + 1, \quad n(G) = n(G') + 1$$

Also $f(G) = f(G') - 1$ because the handle connects two distinct faces in G' making a single face in G . Hence by substitution:

$$F(G) = e(G) - n(G) + 2 - 2g.$$

And so by induction the theorem is proved.

Corollary

If G is a connected graph with genus $n(G) \geq 4$ then

$$g \geq \frac{1}{6} (e(G) - 3n(G)) + 1$$

Proof.

From the above theorem we get:

$$g = \frac{1}{2} (e(G) - n(G) - f(G)) + 1$$

Every face of an embedding of the graph is bound by at least three edges of which separates two faces, therefore $3f(G) \leq 2e(G)$, and so the result follows by substitution.

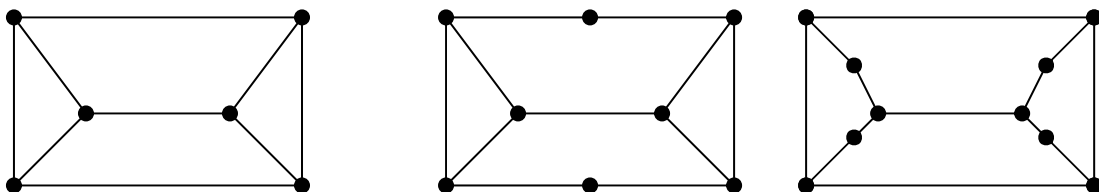
Definition

Homeomorphic graphs

Two graphs are said to be *homeomorphic* if one can be made isomorphic to the other by the addition or the deletion of vertices of degree two, in the following manner:

By dividing an edge into two edges in series by the insertion of a vertex of degree 2, or by reverse of this process;

Example



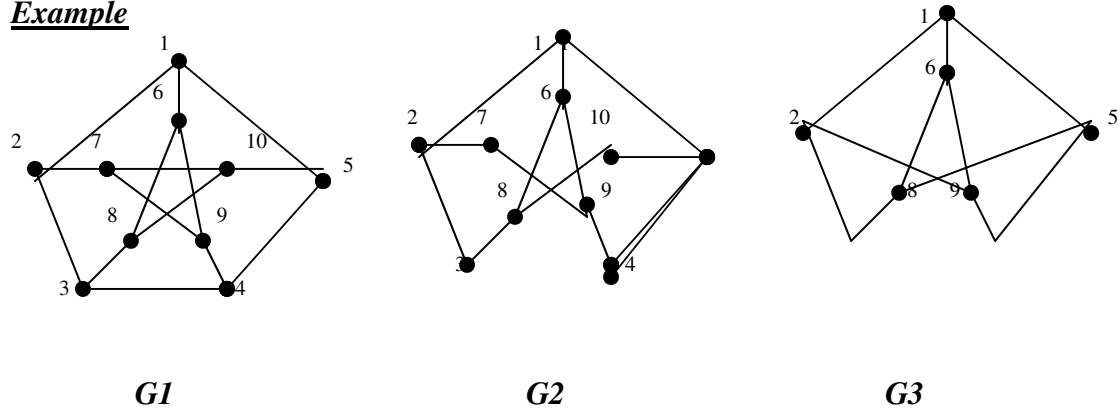
These graphs are homeomorphic

Theorem (Kuratowski)

A graph is planar if and only if it has no sub-graph homeomorphic to K_5 or to $K_{3,3}$.

Proof. The proof is out of the subject of this course. For a proof you can look at Alan Gibbons book, "Algorithmic graph theory", page 77.

Example



Graph G1 is not planar, since it has a sub-graph (G2) homeomorphic to G3, Which is isomorphic to $K_{3,3}$ (The partition of G3 vertices is { 1,8,9} and {2,5,6})

Definitions

Coloring

A *coloring* of the vertices of a graph is a mapping of any vertex of the graph to a color such that any vertices connected with an edge have different colors.

The minimum number of colors required for a graph coloring is called *coloring number* of the graph.

Theorem (Four Color theorem)

A planar graph has coloring number ≤ 4

The proof was given in 1976 by Appel and Haken, using computers and analyzing the problem to 1936 different cases.

Lemma

The four color theorem is equivalent to the following lemma:

The vertices of a planar graph can be partitioned in two sets, V_1, V_2 such that $G/V_1, G/V_2$ are both bipartite graphs.

Proof.

a) *4 Color Theorem* \Rightarrow *Lemma*

I can get immediately the two bipartite graphs by choosing 2 of the four colors for one of the bipartite graphs.

b) *Lemma* \Rightarrow *4 Color theorem*

I can get the two bipartite graphs and put 2 different colors on each graph.

Definition

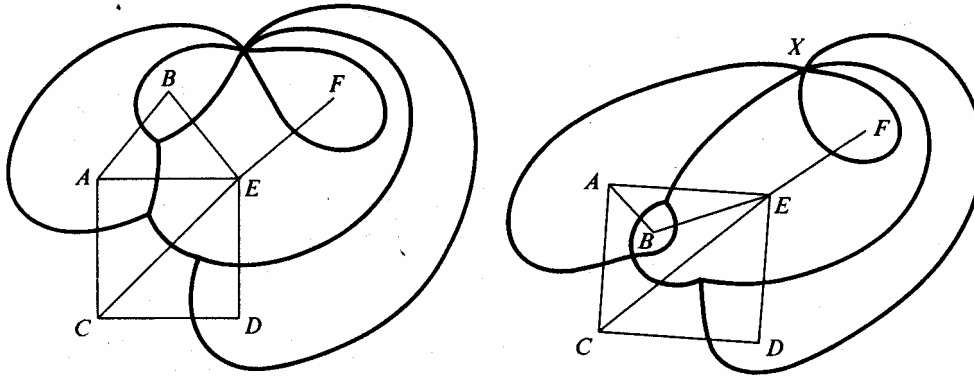
Dual graph.

Let G be a planar graph and a planar embedding G'

A dual graph G^* has a vertex for each face of G' and an edge between two faces f_1 and f_2 if and only if f_1 and f_2 share an edge.

Dual graph is not always a simple graph.

Example



We can see that the dual of the first planar representation of the graph, is not isomorphic to the dual of the second representation.

The dual graph refers to of a planar representation and not to the graph.

Lemma

A graph is planar if and only if the dual graphs of it's planar embeddings are planar

For a proof you can look at Alan Gibbons book, "Algorithmic graph theory", page 83.

Corollary

Using dual graphs, w can see that the four color theorem is equivalent to:

We can color the faces of a graph such that two faces who share an edge have different color, using at most four colors.

Planarity Testing

A sophisticated algorithm given by Hopcroft and Tarjan , tests planarity in $O(n)$ time.

Here we will see a simple algorithm that test planarity $O(n^3)$ time.

As a first application of the divide and conquer principle , we observe that:

- A graph is planar if and only if all its connected components are planar
- A connected graph is planar if and only if all its biconnected components are planar

Thus, via preliminary decomposition into connected and biconnected components, we can restrict our attention to the problem of testing the planarity of a biconnected graph.

Definitions

Piece

Let G be a biconnected graph.

Let C be a cycle in G .

A *piece* is:

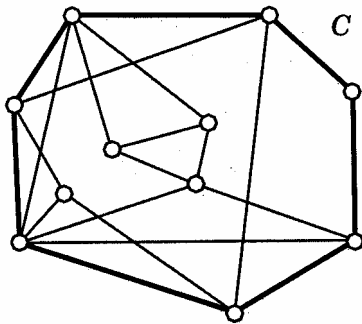
- an edge with vertices on C
- a connected component after removing the edges of C

Separating cycle

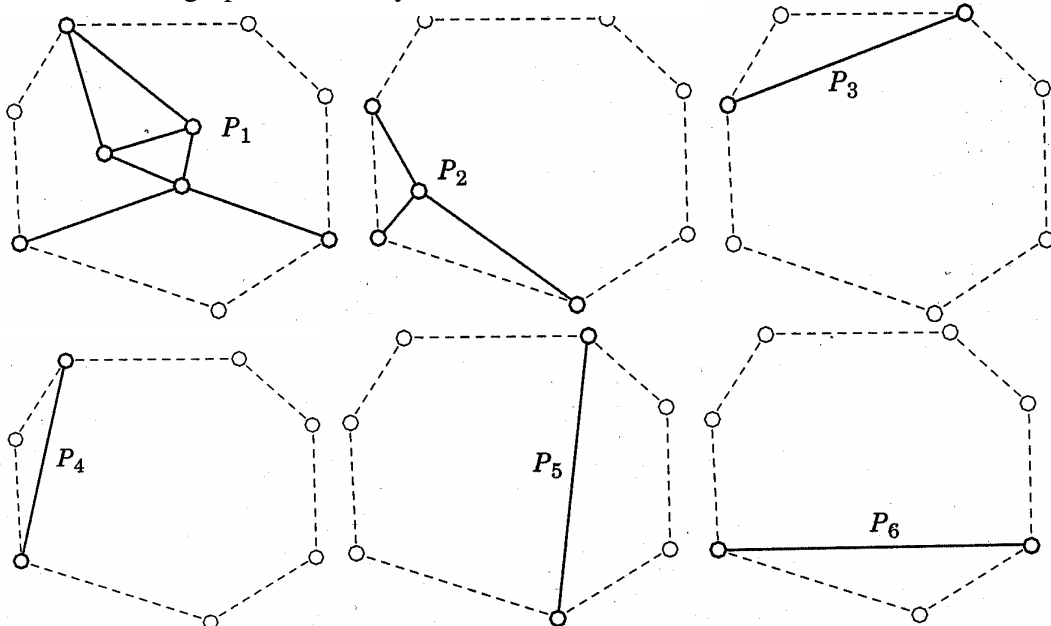
A cycle C of G is said to be *separating* if it has at least two pieces, and is called *nonseparating* if it has one piece. Of course, if $G = C$, then C has no pieces.

In any biconnected graph there is a separating cycle and from any non-separating cycle, I can always get a separating cycle.

Example



A biconnected graph G and a cycle C .



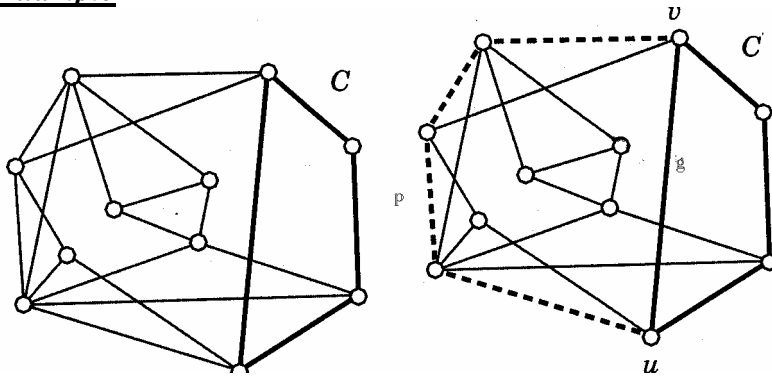
The pieces of G with respect to C

Lemma

Let G be a biconnected graph and let C be a nonseparating cycle of G with piece P . If P is not a path, then G has a separating cycle C' consisting of a subpath of C plus a path of P between two attachments.

Proof. Let u and v be two attachments of P that are consecutive in the circular ordering, and let g be a subpath of C between u and v that does not contain any attachment of C . Since P is connected, there is a path p , between u and v . Let C' be the cycle obtained from C by replacing g with p . We have that g is a piece of G with respect to C' . If P is not a path, let e be an edge of P not in p . There is a piece of C' distinct from g containing e . Thus, if P is not a path, then C' has at least two pieces and is thus a separating cycle of G .

Example



A nonseparating cycle C and a separating cycle C' obtained from C as shown in the proof of the Lemma.

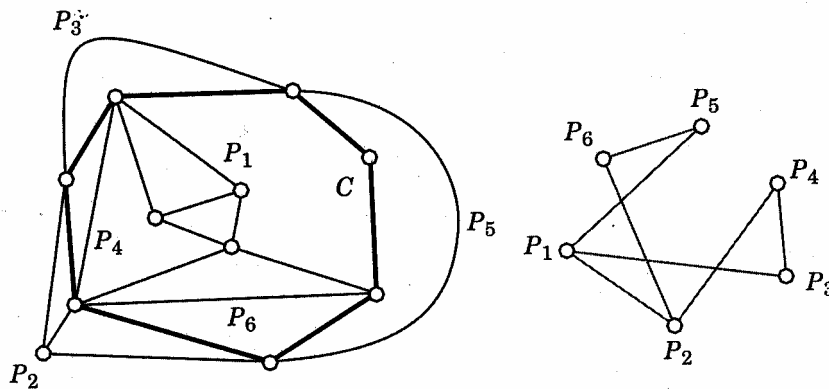
Definition

The *interlacement graph*

If the graph G is planar, then in any planar drawing of G each piece is drawn either inside C or entirely outside C . We say that two pieces of G , with respect to C , *interlace* if they cannot be drawn on the same side of C without violating planarity.

The *interlacement graph* of the pieces of G , with respect to C , is the graph whose vertices are the pieces of G and whose edges are the pairs of pieces that interlace.

Example



A planar drawing of the above graph, where pieces P_1 , P_3 and P_6 are drawn inside cycle C and the other pieces are drawn outside and the interlacement graph I of the pieces of F with respect to cycle C

Theorem

A biconnected graph G with a cycle C is planar if and only if the following conditions hold:

- For each piece P of G with respect to C , the graph obtained by adding P to C is planar
- The interlacement graph of the pieces of G , with respect to C , is bipartite.

Algorithm

The correctness of the algorithm is based on the above lemma and theorem and on the fact that the graph P' obtained by adding a piece P to C is biconnected.

Input: a biconnected graph, G , with n vertices and at most $3n-6$ edges, and a separating cycle C of G .

Output: an indication of whether G is planar.

1. Compute the pieces of G with respect to C .
2. For each piece P of G that is not a path (of one or more edges):
 - (a) let P' be the graph obtained by adding P to C
 - (b) let C' be the cycle of P' obtained from C by replacing the portion of C between two consecutive attachments with a path of P between them
 - (c) apply the algorithm recursively to graph P' and cycle C' . If P' is nonplanar, return “nonplanar”
3. Compute the interlacement graph I of the pieces.
4. Test whether I is bipartite. If it is not bipartite, return “nonplanar”
5. Return “planar”

Time analysis

Step 1: $O(n)$

Step 2: $O(n)$

Step 3: $O(n^2)$

Step 4: $O(n^2)$

So a recursive invocation of algorithm of planarity testing takes $O(n^2)$ time.

We observe that number of recursive invocations is $O(n)$ by associating with each invocation a distinct edge of G . We conclude that the running time of algorithm

Planarity Testing is $O(n^3)$