CS578- Speech Signal Processing

Lecture 1: Discrete-Time Signal Processing Framework

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Outline

Discrete-time Signals

- Unit sample or "impulse":

$$\begin{aligned}\delta[n] &= 1, \quad n = 0 \\ &= 0, \quad n \neq 0\end{aligned}$$

- Unit step:

$$\begin{aligned}u[n] &= 1, \quad n \geq 0 \\ &= 0, \quad n < 0\end{aligned}$$

- Exponential sequence:

$$x[n] = A\alpha^n$$

- Sinusoidal sequence:

$$x[n] = A\cos(\omega n + \phi)$$

- Complex exponential sequence:

$$x[n] = Ae^{j\omega n}$$

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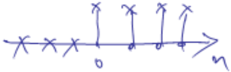

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$\mathrm{Im}\{x[n]\} = A\sin(\omega n + \varphi)$

$\mathrm{Re}\{x[n]\} = A\cos(\omega n + \varphi)$

Discrete-time System:

$$y[n] = T\{x[n]\}$$

Important class of systems: Linear and Time Invariant (LTI):

- Linearity:

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

- Time-Invariant:

$$\begin{aligned} \text{if } y[n] &= T\{x[n]\} \\ \text{then } y[n-n_0] &= T\{x[n-n_0]\} \end{aligned}$$

Important property of LTI:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= x[n] \star h[n] \end{aligned}$$

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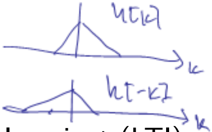

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Stability and Causality for LTI

Necessary and sufficient conditions for:

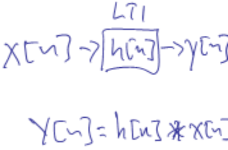
- Stability:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Causality:

$$h[n] = 0, \text{ for } n < 0$$

Stability and Causality for LTI



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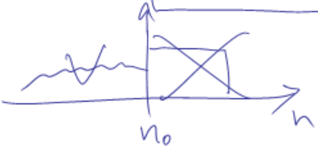

Outline

Discrete-Time Fourier Transform, DTFT

Discrete-Time Fourier Transform pair:

- Direct:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Inverse:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Example:

$$Ae^{j\omega_0 n+\phi} \leftrightarrow 2\pi Ae^{j\phi}\delta(\omega-\omega_0)$$

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$x[n] \overset{F}{\longleftrightarrow} X(\omega)$

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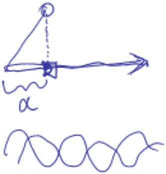


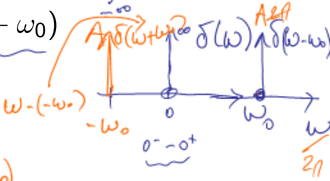
- Inverse:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1$$

Example:

$$A e^{j(\omega_0 n + \phi)} \overset{F}{\longleftrightarrow} 2\pi A e^{j\phi} \delta(\omega - \omega_0)$$



$$F\{A\cos(\omega_0 n + \varphi)\} = F\{A e^{j\varphi} e^{j\omega_0 n}\} + F\{A e^{-j\varphi} e^{-j\omega_0 n}\} =$$

$$= A e^{j\varphi} \cdot 2\pi\, \delta(\omega - \omega_0) + A e^{-j\varphi} 2\pi\, \delta(\omega + \omega_0)$$

$$x[n] = \sum_k A_k \cos(\omega_k n + \varphi_k)$$

DTFT PROPERTIES

- Fourier transform is complex:

$$\begin{aligned} X(\omega) &= X_r(\omega) + jX_i(\omega) \\ &= |X(\omega)|e^{j\angle X(\omega)} \end{aligned}$$

- Fourier transform is periodic with period 2π:

$$X(\omega + 2\pi) = X(\omega)$$

- For real valued sequence $x[n]$:

$$X(\omega) = X^*(-\omega)$$

- Energy of a signal (Parseval theorem):

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

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Uncertainty principle

Given a signal $x[n]$ we define as:

- Duration of the signal:

$$D(x) = \sum_{n=-\infty}^{\infty} (n - \bar{n})^2 |x[n]|^2$$

- Bandwidth of the signal:

$$B(x) = \int_{-\pi}^{\pi} (\omega - \bar{\omega})^2 |X(\omega)|^2 d\omega$$

where

$$\begin{aligned} \bar{n} &= \textstyle\sum_{n=-\infty}^{\infty} n|x[n]|^2 \\ \bar{\omega} &= \textstyle\int_{-\pi}^{\pi} \omega |X(\omega)|^2 d\omega \end{aligned}$$

Uncertainty Principle states that:

$$D(x)B(x) \geq 1/2$$

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Uncertainty Principle states that:

$$D(x)B(x) \geq 1/2$$

$D \cdot B = \alpha$

Hilbert Transform

For a real signal $x[n]$, we form the *analytic signal*:

$$s[n] = s_r[n] + js_i[n]$$

where $s_r[n] = x[n]/2$ and

$$S_i(\omega) = H(\omega)S_r(\omega)$$

where $H(\omega)$ is referred to as *Hilbert transform*:

$$\begin{aligned} H(\omega) &= -j \quad 0 \le \omega < \pi \\ &= j \quad -\pi \le \omega < 0 \end{aligned}$$

Instantaneous amplitude and frequency

The analytic signal may be written as:

$$s[n] = A[n]e^{j\theta[n]}$$

- Instantaneous amplitude:

$$A[n] = |s[n]|$$

- Instantaneous frequency:

$$\omega[n] = \frac{d\theta(t)}{dt}\Big|_{t=nT}$$

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z-Transform

z-Transform pair:

- Direct:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Inverse:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

Example:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

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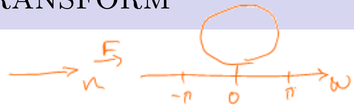

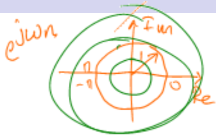

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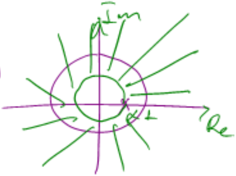

z-Transform: rational functions

Usually:

$$X(z) = \frac{P(z)}{Q(z)} = Az^{-r} \frac{\prod_{k=1}^{M_i}(1 - a_k z^{-1}) \prod_{k=1}^{M_0}(1 - b_k z)}{\prod_{k=1}^{N_i}(1 - c_k z^{-1}) \prod_{k=1}^{N_0}(1 - d_k z)}$$

No repeated poles, no poles outside the unit circle:

$$X(z) = \sum_{k=1}^{N_i} \frac{A_k}{(1 - c_k z^{-1})}$$

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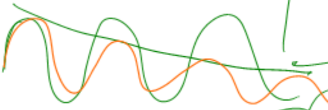


Outline

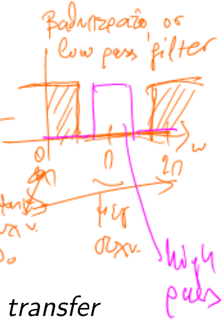
Eigenvalues, Eigenfrequencies, and Eigenfunctions

If $x[n] = e^{j\omega_0 n}$, then

$$y[n] = H(\omega_0)x[n]$$

where $H(\omega)$ is referred to as *frequency response*:

$$H(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$



while $H(z)$ is usually referred to as *system function* or *transfer function*

Convolution theorem

LTI

If

$$x[n] \longleftrightarrow X(\omega)$$
$$h[n] \longleftrightarrow H(\omega)$$

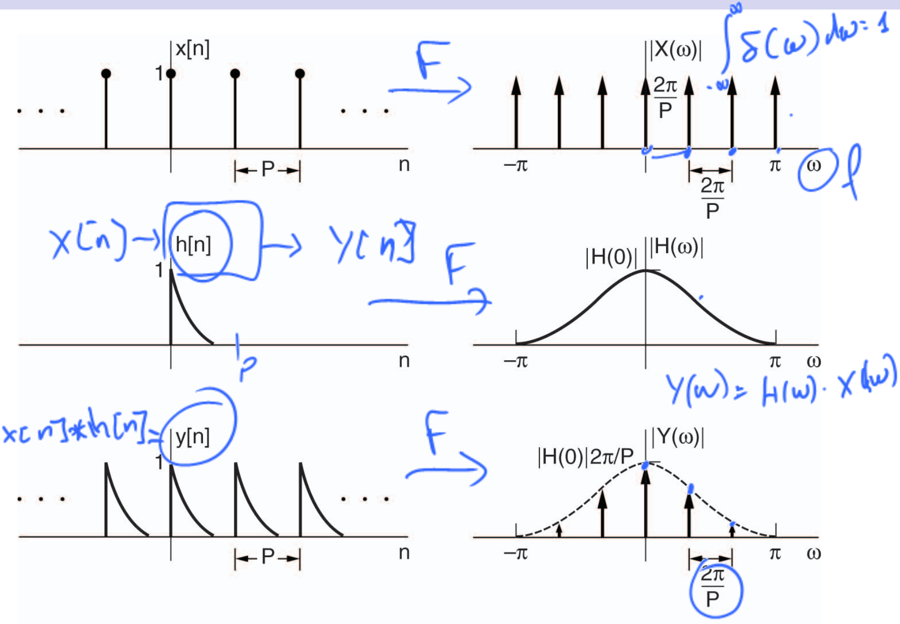
and if: $y[n] = x[n] \star h[n]$, then:

$$Y(\omega) = X(\omega)H(\omega)$$

$j\omega$

$$Y(z) = X(z) \cdot H(z)$$

Example of convolution



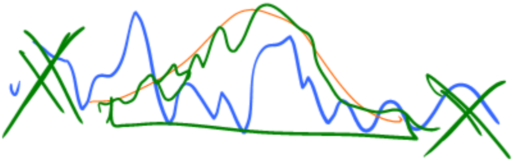
Windowing (Modulation) theorem

If

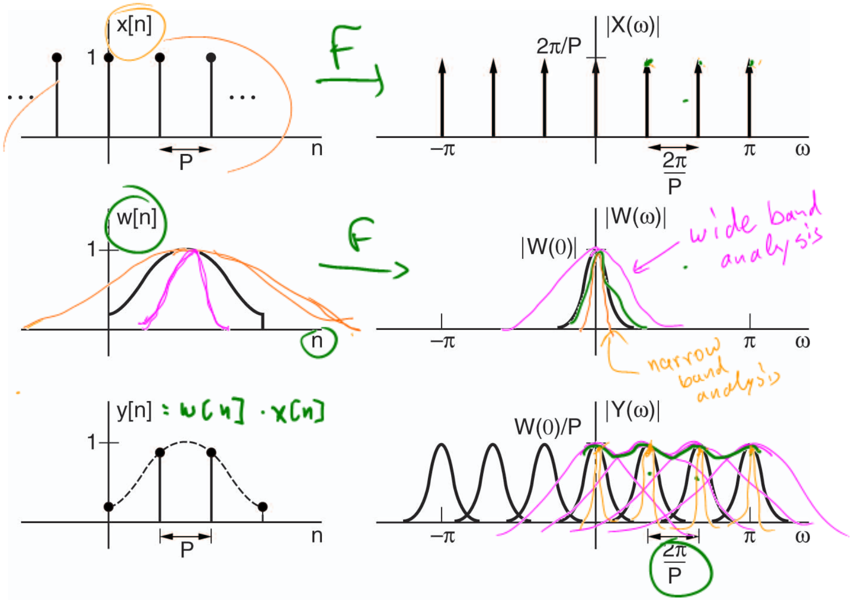
$$x[n] \longleftrightarrow X(\omega)$$
$$w[n] \longleftrightarrow W(\omega)$$

and if: $y[n] = x[n]w[n]$, then:

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Theta)W(\omega-\Theta)d\Theta \\ &= \frac{1}{2\pi}X(\omega) \circledast W(\omega) \end{aligned}$$



Example of modulation



Outline

Difference Equations

In time:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

In z-domain:

$$\begin{aligned} H(z) &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \\ &= A z^{-r} \frac{\prod_{k=1}^{M_i}(1 - a_k z^{-1}) \prod_{k=1}^{M_0}(1 - b_k z)}{\prod_{k=1}^{N_i}(1 - c_k z^{-1})} \end{aligned}$$

Magnitude-Phase Relationships

- Minimum, Maximum and Mixed-phase systems

$$H(z) = H_{min}(z)H_{max}(z)$$

- Minimum-phase and All-pass system

$$H(z) = H_{min}(z)A_{all}(z)$$

Note that

$$\sum_{n=0}^{m} |h_{min}[n]|^2 \geq \sum_{n=0}^{m} |h[n]|^2, \quad m \leq 0$$

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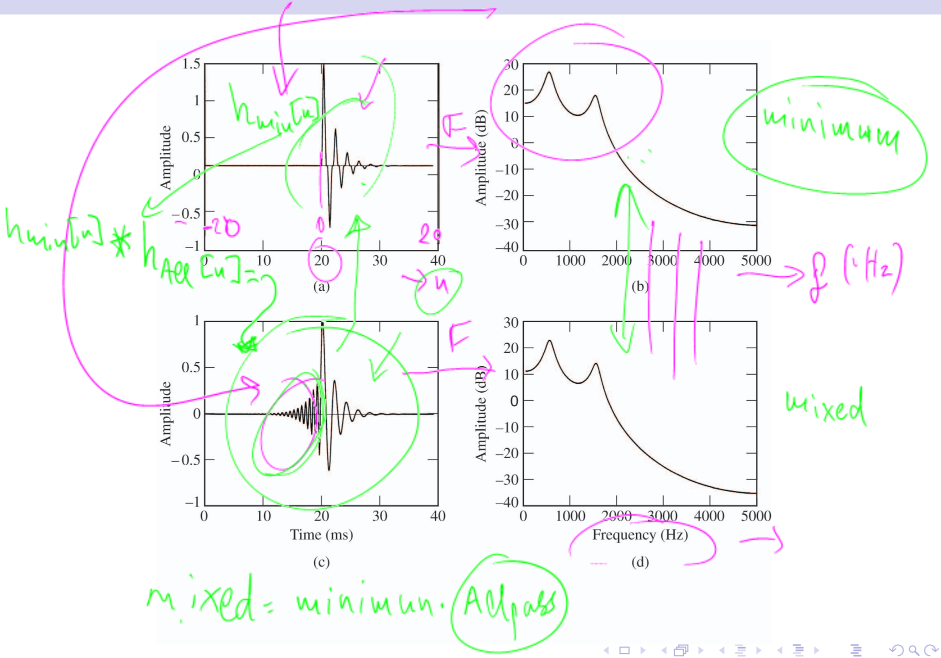
Example of minimum and mixed phase



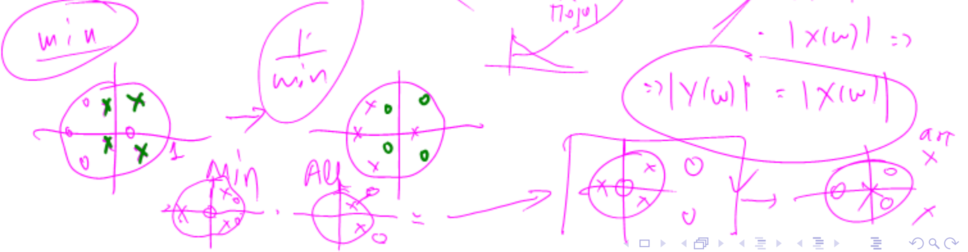
FIR and IIR filters

- FIR:

$$h[n] \neq 0, \quad 0 \leq n < M$$

- IIR:

$$h[n] = \sum_{k=1}^{N_i} A_k c_k^n u[n]$$



Outline

Discrete Fourier Transform

Discrete Fourier Transform, DFT, pair:

- Direct:

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \leq k \leq N-1$$

- Inverse:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad 0 \leq n \leq N-1$$

Parseval theorem:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

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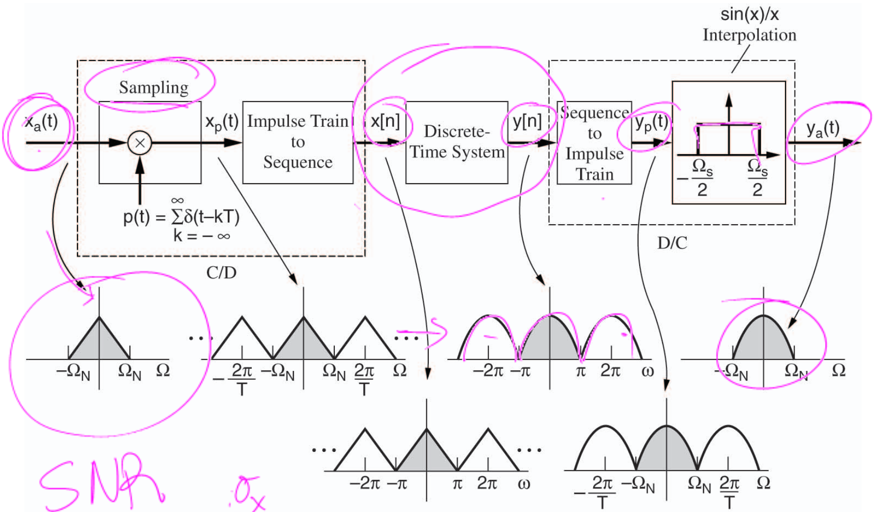
Analog to Digital and Digital to Analog



SNR

$\frac{\sigma_x}{\sigma_\varepsilon}$

$\varepsilon = x - y$