Correspondence

Some Properties of the E Matrix in Two-View Motion Estimation

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Abstract—**In the eight-point linear algorithm for determining 3-D motion/structure from two perspective views using point correspondences, the E matrix occupies a central role. The E matrix is defined as a skew-symmetrical matrix (containing the translation components) postmultiplied by a rotation matrix. In this correspondence, we show that a necessary and sufficient condition for a 3 × 3 matrix to be so decomposable is that one of its singular values is zero and the other two are equal. Several other forms of this property are also presented. Finally, some applications are briefly described.**

Index Terms—**Essential parameters, motion analysis, structure from motion.**

I. Introduction

For determining 3-D motion/structure of a rigid body from two perspective views, a linear algorithm has been discovered by Longuet-Higgins [1] and Tsai and Huang [2]. A more robust version of the algorithm has been recently developed by Faugeras *et al.* [3] and Weng *et al.* [4].

A central concept of the linear algorithm is the 3 × 3 matrix

$$E = \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix} \tag{1}$$

which is defined as

$$E = TR \tag{2}$$

where R is a rotation matrix, and

$$T = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix} \tag{3}$$

is a skew-symmetrical matrix containing the elements of the translation vector (t_1, t_2, t_3). It can readily be shown [2] that the image coordinates (x, y) and (x', y') of a point correspondence between the two views satisfy

$$[x'y'1]E\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \tag{4}$$

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which is linear and homogeneous in the e_i's. The linear algorithm consists of solving for the e_i's in (4) from eight or more point correspondences, and then determine T and R from E [1]–[4].

The purpose of the present paper is to derive some interesting and useful properties of the E matrix. Some applications of these properties are described in Section VI.

II. A Conjecture Proved

We call a 3 × 3 matrix "decomposable" if and only if it can be expressed as a skew-symmetrical matrix postmultiplied by a rotation matrix. Thus, E in (2) is decomposable. In [2], it was shown that a necessary condition for a matrix to be decomposable is that one of its singular values is zero and the other two singular values are equal. Recently, Braccini made the conjecture that this condition is also sufficient [5]. We shown that his conjecture is, in fact, correct.

Theorem: A 3 × 3 matrix B is decomposable if and only if one of its singular values is zero and the other two are equal.

Proof:

Necessity: Assume B is decomposable:

$$B = TR$$

where T is skew symmetrical and R is a rotation. Then we can find an orthonormal matrix Q such that

$$T = Q^t \begin{bmatrix} 0 & \phi & 0 \\ -\phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q \tag{5}$$

where ϕ is a real constant. Thus,

$$B^tB = R^tT^tTR = (QR)^t \begin{bmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} (QR). \tag{6}$$

The singular values of B are therefore $0, \phi^2, \phi^2$.

Sufficiency: Assume one of the singular values of B is zero and the other two are equal. Then

$$BB^t = P^t \begin{bmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P \tag{7}$$

where P is orthonormal and can be chosen as a rotation.

From (7),

$$AA^t = \begin{bmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

where

$$A = PB. \tag{9}$$

Let

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{10}$$

where a_1, a_2, a_3 are the row vectors of A. Then from (8),

$$\begin{aligned} a_1 \cdot a_1 &= \phi^2 & a_1 \cdot a_2 &= 0 \\ a_2 \cdot a_2 &= \phi^2 & a_2 \cdot a_3 &= 0 \\ a_3 \cdot a_3 &= 0 & a_3 \cdot a_1 &= 0 \end{aligned}$$

whence

$$A = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$$

where

$$a_1 \cdot a_1 = \phi^2 = a_2 \cdot a_2$$
$$a_1 \cdot a_2 = 0.$$

Let

$$A' \triangleq \begin{bmatrix} a_1 \\ a_2 \\ a_3' \end{bmatrix} \tag{11}$$

where

$$a_3' = a_1 \times a_2 / \phi. \tag{12}$$

Then A'/ϕ is a rotation (we assume $\phi > 0$). Note that

$$\begin{aligned} A(A')^t = AA^t &= \begin{bmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{13}$$

whence

$$A = \begin{bmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A'. \tag{14}$$

Finally, from (9),

$$\begin{aligned} B = P^tA &= P^t \begin{bmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A' \\ &= P^t \begin{bmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} PP^t \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A' \\ &= TR \end{aligned} \tag{15}$$

where

$$T \triangleq P^t \begin{bmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P\phi$$

is skew-symmetrical and

$$R \triangleq P^t \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A'/\phi$$

is a rotation.

III. First Set of Polynomial Conditions

Let

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{16}$$

where b_1, b_2, b_3 are the row vectors of B. Then, it is obvious that the condition "one of the singular values of B is zero" is equivalent to

$$b_1 \cdot b_2 \times b_3 = 0. \tag{17}$$

It is less obvious, but nonetheless verifiable after a lengthy derivation, that the condition "the other two singular values of B are equal" is equivalent to

$$\begin{aligned} &\|b_1 \times b_2\|^2 + \|b_2 \times b_3\|^2 + \|b_3 \times b_1\|^2 \\ &= \tfrac{1}{4}(\|b_1\|^2 + \|b_2\|^2 + \|b_3\|^2)^2. \end{aligned} \tag{18}$$

IV. Second Set of Polynomial Conditions

It has been discovered by Longuet-Higgins [6] that if B is decomposable, then

$$\frac{b_1}{b_2 \cdot b_3} + \frac{b_2}{b_3 \cdot b_1} + \frac{b_3}{b_1 \cdot b_2} = 0, \tag{19}$$

which gives three scalar equations.

This result can be derived by using the fact that: if

$$B = TR,$$

then,

$$[t_1, t_2, t_3]B = 0 \tag{20}$$

and

$$BB^t = -T^2. \tag{21}$$

V. Third Set of Polynomial Conditions

Huang and Shim [7] showed that if B is decomposable, then

$$b_1 \cdot b_2 \times b_3 = \tag{22}$$

$$(\|b_3\|^2 + \|b_2\|^2 - \|b_1\|^2)(b_2 \cdot b_3) + (b_1 \cdot b_2)(b_1 \cdot b_3) = 0 \tag{23}$$

and

$$\|b_3\|^4 = (\|b_2\|^2 - \|b_1\|^2)^2 + 4(b_1 \cdot b_2)^2. \tag{24}$$

VI. Resolution of an Apparent Contradiction

The singular value result in Section II gives two conditions, which can be written as two polynomial equations in the components of B, e.g., (17) and (18). On the other hand, the result in Sections IV or V gives three equations. How do we resolve this apparent contradiction?

Prof. Longuet-Higgins pointed out [8] that generally for a symmetrical matrix to have equal eigenvalues, two or more conditions have to hold among its elements. In the 2×2 case, it can be easily verified that the matrix $[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}]$ will have equal eigenvalues iff $a = c$ and $b = 0$. In our case, we have a 3×3 symmetrical matrix BB^t. However, one of the eigenvalues is zero. Therefore, by a proper rotation of the coordinate system, we can reduce the situation of the 2×2 case. The condition of equal eigenvalues then becomes two conditions.

We note that recently it has been proven [9] that whereas the equation set (17) and (18) is necessary and sufficient for the decomposability of B, the equation sets (19) and (22)–(24) are necessary, but not sufficient.

VII. Applications

Because of noise in the image coordinates, the E matrix determined from (4) will not be exactly decomposable. This may introduce large errors in the estimation of R and T. Braccini *et al.* [10] have shown experimentally that better estimation accuracy can be achieved if one imposes the decomposability conditions while solving for E from (4).

The linear algorithm for motion estimation described briefly in Section I fails in certain degenerate cases. In particular, it fails if the number of point correspondences given is less than eight. Huang and Shim [7] showed that by imposing the decomposability conditions of E, the linear algorithm can be resurrected.

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References

[1] H. C. Longuet-Higgins, "A computer program for reconstructing a scene from two projections," *Nature*, vol. 392, pp. 133–135, 1981.
[2] R. Y. Tsai and T. S. Huang, "Uniqueness and estimation of 3-D motion parameters of rigid bodies with curved surfaces," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. PAMI-6, pp. 13–27, Jan. 1984.
[3] O. D. Faugeras, F. Lustman, and G. Toscani, "Motion and structure from motion from point and line matches," in *Proc. 1st ICCV*, London, England, June 1987.
[4] J. Weng, T. S. Huang, and N. Ahuja, "Error analysis of motion parameters determination from image sequences," in *Proc. 1st ICCV*, London, England, June 1987.
[5] C. Braccini, private communication to O. D. Faugeras, Feb. 1987.
[6] H. C. Longuet-Higgins, private communication to T. S. Huang, Jan. 29, 1987.
[7] T. S. Huang and Y. S. Shim, "Linear algorithm for motion estimation: How to handle degenerate cases," Coord. Sci. Lab., Univ. Illinois, Urbana, Tech. Note ISP-102, Nov. 30, 1986.
[8] H. C. Longuet-Higgins, private communication to T. S. Huang, Mar. 23, 1987.
[9] A. Netravali, T. S. Huang, R. Holt, and A. S. Krishnakumar, "Algebraic methods in 3-D motion estimation from two-view point correspondences," *Int. J. Imaging Syst. Technol.*, to be published.
[10] C. Braccini, G. Gambardella, A. Grattarola, and S. Zappatore, "Motion estimation of rigid bodies: Effects of the rigidity constraints," in *Proc. EUSIPCO-86*, The Hague, The Netherlands, Sept. 1986.

Computational Feasibility of Structured Matching

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***Abstract*—Structured matching is a task-specific technique for selecting one choice out of a small number of alternatives based on a given set of parameters. In structured matching, the knowledge and control for making a decision are integrated within a hierarchical structure. Each node in the hierarchy corresponds to a different aspect of the decision and contains knowledge for directly mapping the results of its children nodes (or selected parameters) into a choice on the subdecision. The root node selects the final choice for the decision. We formally characterize the task and strategy of structured matching and analyze its computational complexity. Structured matching, we believe, captures the essence of what makes a range of decision-making problems computationally feasible to solve.**

***Index Terms*—Artificial intelligence, computational complexity, decision making, hierarchical systems, knowledge-based systems, task-specific techniques.**

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I. Introduction

Task-specific information processing techniques are beginning to gain significant attention in research on artificial intelligence [4], [8], [12]. A task-specific technique solves a limited range of problems by applying a computationally tractable method to a highly constrained knowledge representation. If a problem can be adequately solved by a task-specific technique, then the technique can provide useful constraints on knowledge engineering, explanation, learning, cognitive modeling, etc.

In this correspondence, we consider a task-specific technique called *structured matching* [1]. Structured matching is applicable to decision-making tasks that involve selecting one choice out of a small number of alternatives based on a given set of parameters. In structured matching, the knowledge and control needed for making such decisions are integrated within a hierarchical structure. Each node in the hierarchy corresponds to a different aspect of the decision and contains knowledge for directly mapping the results of its children nodes (or selected parameters) into a choice on the subdecision. The root node selects the final choice for the decision.

Below, we first describe structured matching informally, followed by a formal definition of the task and strategy of structured matching. Next, we analyze the computational complexity of structured matching, making explicit the conditions under which the technique is tractable. Structured matching we believe, captures the essence of what makes a range of decision-making problems feasible to solve.

II. Selecting One Out of Several Choices

All knowledge-based systems need to make decisions, e.g., perform classification or recognition. Often, these decisions have three characteristics: 1) a small number of alternatives to select from, 2) several parameters (including perhaps previous decisions) that are known to be relevant to the decision where each parameter can take a value from a small number of values, and 3) the need to make the same decision over and over again in different situations. For instance, a diagnostic system needs to rate the confidence of a malfunction, e.g., whether the malfunction is definite, probable, possible, or ruled out. A diagnostic system also needs to decide what malfunction(s) should be considered. Typically, to make these decisions, several pieces of evidence, such as the presence or absence of features and previous decisions about malfunctions, need to be taken into account. If evaluating the presence of the malfunction is a routine part of the system's problem solving, then it is useful to explicitly encode the knowledge for making the decision.

A. Simple Matching

One way to encode knowledge for making a decision is to directly associate patterns of parameters with the choices, for example,

$$prmtr1 \wedge prmtr2 \wedge (prmtr3 \vee prmtr4) \Rightarrow malfunction1 \textit{ is Probable}.$$