

Lecture 11: The Simply Typed Lambda-Calculus

Additional types

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Last time

- The unit type
- Pairs, the product type $T \times T$
- Binary unions, the sum type $T + T$



The unit type

- Syntax

$$\begin{aligned} e &::= \dots \mid () \mid e_1; e_2 \\ v &::= \dots \mid () \\ T &::= \dots \mid Unit \end{aligned}$$

- Typing

$$\begin{array}{c} [T\text{-UNIT}] \frac{}{\Gamma \vdash () : Unit} \qquad [T\text{-SEQ}] \frac{\Gamma \vdash e_1 : Unit \quad \Gamma \vdash e_2 : T}{\Gamma \vdash e_1; e_2 : T} \end{array}$$

- Semantics

$$\frac{}{(); e_2 \rightarrow e_2} \qquad \frac{e_1 \rightarrow e'_1}{e_1; e_2 \rightarrow e'_1; e_2}$$


Pairs: the product type

- Syntax

$$\begin{aligned}e &::= \dots \mid e.1 \mid e.2 \mid (e, e) \\v &::= \dots \mid (v, v) \\T &::= \dots \mid T \times T\end{aligned}$$

- Typing

$$[\text{T-PAIR}] \frac{\Gamma \vdash e_1 : T_1 \quad \Gamma \vdash e_2 : T_2}{\Gamma \vdash (e_1, e_2) : T_1 \times T_2}$$

$$[\text{T-FST}] \frac{\Gamma \vdash e : T \times T'}{\Gamma \vdash e.1 : T} \quad [\text{T-SND}] \frac{\Gamma \vdash e : T \times T'}{\Gamma \vdash e.2 : T'}$$



Pairs: the product type (cont'd)

- Semantics

$$\frac{}{(v_1, v_2).1 \rightarrow v_1} \qquad \frac{}{(v_1, v_2).2 \rightarrow v_2}$$
$$\frac{e_1 \rightarrow e'_1}{(e_1, e_2) \rightarrow (e'_1, e_2)} \qquad \frac{e_2 \rightarrow e'_2}{(v_1, e_2) \rightarrow (v_1, e'_2)}$$
$$\frac{e \rightarrow e'}{e.1 \rightarrow e'.1} \qquad \frac{e \rightarrow e'}{e.2 \rightarrow e'.2}$$



Binary unions: the sum type

- Syntax

$$\begin{aligned} e &::= \dots \mid \text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of inl } x \Rightarrow e \mid \text{inr } x \Rightarrow e \\ v &::= \dots \mid \text{inl } v \mid \text{inr } v \\ T &::= \dots \mid T + T \end{aligned}$$

- Typing

$$\begin{array}{c} \text{[T-INL]} \frac{\Gamma \vdash e : T_1}{\Gamma \vdash \text{inl } e : T_1 + T_2} \qquad \text{[T-INR]} \frac{\Gamma \vdash e : T_2}{\Gamma \vdash \text{inr } e : T_1 + T_2} \end{array}$$
$$\text{[T-CASE-UNION]} \frac{\begin{array}{c} \Gamma \vdash e : T_1 + T_2 \\ \Gamma, x_1 : T_1 \vdash e_1 : T \\ \Gamma, x_2 : T_2 \vdash e_2 : T \end{array}}{\Gamma \vdash \text{case } e \text{ of inl } x_1 \Rightarrow e_1 \mid \text{inr } x_2 \Rightarrow e_2 : T}$$


Binary unions: the sum type (cont'd)

- Semantics

$$\frac{e \rightarrow e'}{\text{inl } e \rightarrow \text{inl } e'} \qquad \frac{e \rightarrow e'}{\text{inr } e \rightarrow \text{inr } e'}$$

$$\frac{e \rightarrow e'}{\text{case } e \text{ of inl } x_1 \Rightarrow e_1 \mid \text{inr } x_2 \Rightarrow e_2 \rightarrow \text{case } e' \text{ of inl } x_1 \Rightarrow e_1 \mid \text{inr } x_2 \Rightarrow e_2}$$

$$\frac{}{\text{case } (\text{inl } v) \text{ of inl } x_1 \Rightarrow e_1 \mid \text{inr } x_2 \Rightarrow e_2 \rightarrow e_1[v/x_1]}$$

$$\frac{}{\text{case } (\text{inr } v) \text{ of inl } x_1 \Rightarrow e_1 \mid \text{inr } x_2 \Rightarrow e_2 \rightarrow e_2[v/x_2]}$$



Let bindings

- Syntax

$$e ::= \dots \mid \text{let } x = e \text{ in } e$$

- Typing

$$[\text{T-LET}] \frac{\Gamma \vdash e_1 : T_1 \quad \Gamma, x : T_1 \vdash e_2 : T_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : T_2}$$

- Semantics

$$\frac{}{\text{let } x = v \text{ in } e \rightarrow e[v/x]}$$
$$\frac{e_1 \rightarrow e'_1}{\text{let } x = e_1 \text{ in } e_2 \rightarrow \text{let } x = e'_1 \text{ in } e_2}$$



Let bindings (cont'd)

- In untyped calculus defined as syntactic sugar (preprocessed):

$$\text{let } x = e_1 \text{ in } e_2 \doteq (\lambda x. e_2) e_1$$

- Now, functions are annotated with types:

$$\text{let } x = e_1 \text{ in } e_2 \doteq (\lambda x : T_1. e_2) e_1$$

- What is the type T_1 ?
- Answer: ask the typechecker!
- Let-binding not exactly syntactic sugar in typed calculus



Tuples

- Generalize pairs to tuples of n items
- Syntax

$$\begin{aligned}e &::= \dots \mid e.i \mid \{e_1, \dots, e_n\} \\v &::= \dots \mid \{v_1, \dots, v_n\} \\T &::= \dots \mid \{T_1 \times \dots \times T_n\}\end{aligned}$$

- Typing

$$[\text{T-TUPLE}] \frac{\forall 1 \leq i \leq n, \Gamma \vdash e_i : T_i}{\Gamma \vdash \{e_1, \dots, e_n\} : \{T_1 \times \dots \times T_n\}}$$

$$[\text{T-PROJ}] \frac{\Gamma \vdash e : \{T_1 \times \dots \times T_n\}}{\Gamma \vdash e.i : T_i}$$



Tuples (cont'd)

- Semantics

$$\frac{\frac{e \rightarrow e'}{e.i \rightarrow e'.i} \quad \frac{}{\{v_1, \dots, v_n\}.i \rightarrow v_i}}{e_j \rightarrow e'_j}}{\{v_1, \dots, v_{j-1}, e_j, \dots, e_n\} \rightarrow \{v_1, \dots, v_{j-1}, e'_j, \dots, e_n\}}$$



Records

- Generalize tuples with arbitrary labels: structs
- Syntax

$$e ::= \dots \mid e.l \mid \{l_1 = e_1, \dots, l_n = e_n\}$$

$$v ::= \dots \mid \{l_1 = v_1, \dots, l_n = v_n\}$$

$$T ::= \dots \mid \{l_1 : T_1, \dots, l_n : T_n\}$$

- Typing

$$[\text{T-RECORD}] \frac{\forall 1 \leq i \leq n, \Gamma \vdash e_i : T_i}{\Gamma \vdash \{l_1 = e_1, \dots, l_n = e_n\} : \{l_1 : T_1, \dots, l_n : T_n\}}$$

$$[\text{T-PROJ}] \frac{\Gamma \vdash e : \{l_1 : T_1, \dots, l_n : T_n\}}{\Gamma \vdash e.l_i : T_i}$$



Records (cont'd)

- Semantics

$$\frac{e \rightarrow e'}{e.l_i \rightarrow e'.l_i} \quad \frac{}{\{l_1 = v_1, \dots, l_n = v_n\}.l_i \rightarrow v_i}$$
$$\frac{e_j \rightarrow e'_j}{\{l_1 = v_1, \dots, l_{j-1} = v_{j-1}, l_j = e_j, \dots, l_n = e_n\} \rightarrow \{l_1 = v_1, \dots, l_{j-1} = v_{j-1}, l_j = e'_j, \dots, l_n = e_n\}}$$



Variants

- Generalization of unions (sums)
- Syntax

$$\begin{aligned} e &::= \dots \mid l \langle e \rangle \text{ as } T \mid \text{case } e \text{ of } \{l_i \langle x_i \rangle \Rightarrow e_i\}^{1 \leq i \leq n} \\ v &::= \dots \mid l \langle v \rangle \text{ as } T \\ T &::= \dots \mid \langle l_i : T_i^{1 \leq i \leq n} \rangle \end{aligned}$$

- Typing

$$\text{[T-VARIANT]} \frac{\Gamma \vdash e_j : T_j}{\Gamma \vdash l_j \langle e_j \rangle \text{ as } \langle l_i : T_i^{1 \leq i \leq n} \rangle : \langle l_i : T_i^{1 \leq i \leq n} \rangle}$$

$$\text{[T-CASE-VAR]} \frac{\Gamma \vdash e : \langle l_i : T_i^{1 \leq i \leq n} \rangle \quad \forall 1 \leq i \leq n, \Gamma, x_i : T_i \vdash e_i : T}{\Gamma \vdash \text{case } e \text{ of } \{l_i \langle x_i \rangle \Rightarrow e_i\}^{1 \leq i \leq n} : T}$$



Variants (cont'd)

- Semantics

$$\frac{}{\text{case } (l_j \langle v_j \rangle \text{ as } T) \text{ of } \{l_i \langle x_i \rangle \Rightarrow e_i\}^{1 \leq i \leq n} \rightarrow e_j[v_j/x_j]}$$

$$\frac{e \rightarrow e'}{\text{case } e \text{ of } \{l_i \langle x_i \rangle \Rightarrow e_i\}^{1 \leq i \leq n} \rightarrow \text{case } e' \text{ of } \{l_i \langle x_i \rangle \Rightarrow e_i\}^{1 \leq i \leq n}}$$

$$\frac{e_i \rightarrow e'_i}{l_i \langle e_i \rangle \text{ as } T \rightarrow l_i \langle e'_i \rangle \text{ as } T}$$



Lists

- Syntax

$$\begin{aligned} e &::= \dots \mid \text{nil } [T] \mid \text{cons } [T] e e \mid \text{case } e \text{ of } \{ \text{nil} \Rightarrow e \mid \text{cons } x x \Rightarrow e \} \\ v &::= \dots \mid \text{nil } [T] \mid \text{cons } [T] v v \\ T &::= \dots \mid \text{List } T \end{aligned}$$

- Typing

$$\begin{array}{c} \text{[T-NIL]} \frac{}{\Gamma \vdash \text{nil } [T] : \text{List } T} \qquad \text{[T-CONS]} \frac{\Gamma \vdash e : T \quad \Gamma \vdash e' : \text{List } T}{\Gamma \vdash \text{cons } [T] e e' : \text{List } T} \\ \\ \text{[T-CASE-LIST]} \frac{\Gamma \vdash e : \text{List } T \quad \Gamma \vdash e_1 : T' \quad \Gamma, x_h : T, x_t : \text{List } T \vdash e_2 : T'}{\Gamma \vdash \text{case } e \text{ of } \{ \text{nil} \Rightarrow e_1 \mid \text{cons } x_h x_t \Rightarrow e_2 \} : T'} \end{array}$$



Lists (cont'd)

- Semantics

$$\frac{e_1 \rightarrow e'_1}{\text{cons } [T]e_1 \ e_2 \rightarrow \text{cons } [T]e'_1 \ e_2} \qquad \frac{e_2 \rightarrow e'_2}{\text{cons } [T]v \ e_2 \rightarrow \text{cons } [T]v \ e'_2}$$

$$\frac{e \rightarrow e'}{\text{case } e \text{ of } \{\text{nil} \Rightarrow e_1 \mid \text{cons } x_h \ x_t \Rightarrow e_2\} \rightarrow \text{case } e' \text{ of } \{\text{nil} \Rightarrow e_1 \mid \text{cons } x_h \ x_t \Rightarrow e_2\}}$$
$$\frac{}{\text{case nil } [T] \text{ of } \{\text{nil} \Rightarrow e_1 \mid \text{cons } x_h \ x_t \Rightarrow e_2\} \rightarrow e_1}$$

$$\frac{}{\text{case cons } [T]v_1 \ v_2 \text{ of } \{\text{nil} \Rightarrow e_1 \mid \text{cons } x_h \ x_t \Rightarrow e_2\} \rightarrow e_2[v_1/x_h, v_2/x_t]}$$

