Section 8
Leader Election in Rings

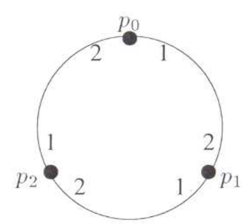
The Leader Election Problem

- Each process should eventually decide that it is either the leader or it is not the leader.
- Exactly one process should decide that it is the leader.
- The leader process may be responsible for achieving synchronization in future activities of the system:
- token re-creation
- recovery from deadlock
- play the role of the root node in the construction of a spanning tree, etc.

The Leader Election Problem – More formally

- An algorithm is said to solve the leader election problem if it satisfies the following conditions:
  - The terminated states are partitioned into elected and not-elected states. Once a process enters an elected (respectively, not-elected) state, its transition function will only move it to another (or the same) elected (respectively, not-elected) state.
  - In every admissible execution, exactly one process (the leader) enters an elected state and all the remaining processes enter a not-elected state.

The Leader Election Problem



Assumptions

- Ring topology
- The n processes have
  notion of left and righ
  - For every i, $1 \leq i \leq n$, p_i's channel to p_{i+1} is labeled 1, also known as left or clock-wise, and p_i's channel to p_{i-1} is labeled 2, also known as right or counter-clock-wise (addition and subtraction here are modulo n).

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Model - Rings

- An algorithm is anonymous if the processes do not have unique identifiers that can be used by the algorithm.
  - Every process has the same state machine.
- Otherwise, the algorithm is called eponymous (or non-anonymous).
- If n is not known to the algorithm, the algorithm is called uniform
  - The algorithm looks the same for every value of n.
- In an anonymous non-uniform algorithm, for each value of n, there is a single state machine, but there can be different state machines for different ring sizes.
  - n can be explicitly present in the code.

Leader Election in Anonymous Synchronous Rings

Theorem: There is no non-uniform anonymous algorithm for leader election in synchronous rings.

Lemma: For every round k of the admissible execution of an anonymous leader election algorithm in a ring, the states of all the processors at the end of round k are the same.

Proof: By induction on k.

- **Base case:** Straightforward since all processes begin in the same state.
- **Induction Hypothesis:** Assume the lemma holds for round k-1.
- **Induction Step:** Since all processes are in the same state in round k-1, they all send the same messages m_l to the left and m_r to the right.
- In round k, all processes receive message m_r on its left edge and m_l on its right; because they execute the same program, they are in the same state at the end of round k.

Leader Election in Eponymous Asynchronous Rings

An $O(n^2)$ Algorithm

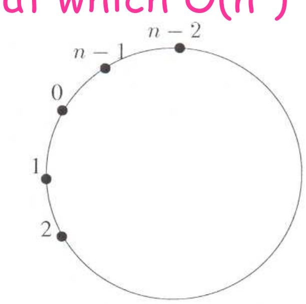
Description of the algorithm:

- Each process sends a message with its identifier to its left neighbor and then waits for messages from its right neighbor.
- When is receives such a message, it checks the identifier in the message:
  - If it is greater than its own identifier, it forwards the message to the left.
  - Otherwise, it shallows the message.
- If a processor receives a message with its own identifier, it declares itself a leader by sending a termination message to its left neighbor and terminating.
- A processor that receives the termination message, forwards it to the left and terminates as non-leader.

Leader Election in Eponymous Asynchronous Rings

Communication Complexity?

- No process sends more than n messages.
- Is there an execution at which $\Theta(n^2)$ messages are sent?



An Algorithm with Communication Complexity O(nlogn) – Main Ideas

- The k-neighborhood of a process p_i in the ring is the set of processes that are at distance at most k from p_i in the ring (either to the left or to the right).

Main Ideas

- The algorithm works in phases:
  - k^{th} phase, $k \geq 0$: a process tries to become a winner for the phase; a process becomes a winner if it has the largest id in its 2^k-neighborhood.
  - Only processes that are winners in the k^{th} phase continue to compete in the $(k+1)^{st}$ phase.

An Algorithm with Communication Complexity O(nlogn)- Description

- In phase k, a process p_i that is a phase k-1 winner sends <probe> messages with its identifier to the 2^k-neighborhood (one in each direction).
- A <probe> is shallowed by a processor if it contains an identifier that is smaller than its own identifier.
- If the message arrives at the last process in the neighborhood, then that last process sends back a <reply> message to p_i.
- If p_i receives replies from both directions, it becomes a phase k winner, and it continues to phase k+1.
- A processor that receives its own <probe> message terminates the algorithm as the leader and sends a termination message around the ring.

An Algorithm with Communication Complexity O(nlogn) - Pseudocode

Algorithm 5 Asynchronous leader election: code for processor p_i, $0 \leq i < n$.

```
Initially, asleep = true

upon receiving no message:
    if asleep then
        asleep := false
        send ⟨probe,id,0,1⟩ to left and right

upon receiving ⟨probe,j,k,d⟩ from left (resp., right):
    if j = id then terminate as the leader
    if j > id and d < 2^k then                        // forward the message
        send ⟨probe,j,k,d + 1⟩ to right (resp., left)  // increment hop counter
    if j > id and d ≥ 2^k then                        // reply to the message
        send ⟨reply,j,k⟩ to left (resp., right)
                                          // if j < id, message is swallowed

upon receiving ⟨reply,j,k⟩ from left (resp., right):
    if j ≠ id then send ⟨reply,j,k⟩ to right (resp., left)   // forward the reply
    else                                                  // reply is for own probe
        if already received ⟨reply,j,k⟩ from right (resp., left) then
            send ⟨probe,id,k + 1,1⟩ to left and right !   // phase k winner
```

- A message of type <probe> contains the id j of the process that sends it, the phase number k and a hop counter d.
- A message of type <reply> contains the id j and the number of the current phase k.

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An Algorithm with Communication Complexity O(nlogn) - Analysis

- **Lemma:** For each $k \geq 0$, the number of processes that are phase k winners is at most $n/(2^k+1)$.
- **Proof:**
  - Between two winners of phase k there are 2^k other processes in the ring.
- **Remarks**
  - There is just one winner after log(n-1) phases.
  - The total number of messages is:
  - 5n + Sum_{k=1}^{\lceil log(n-1) \rceil+1} $4*2^k*n/(2^{k-1}+1)$ < 5n + 8n(logn+2)
- **Theorem:** There is an asynchronous leader election algorithm whose message complexity is O(nlogn).

Leader Election in Synchronous Rings

- The reception of no message in a round is a piece of information. Does this help?

An O(n) Upper Bound

The Non-Uniform Algorithm

- Elects the processor with the minimal identifier as the leader.
- It works in phases, each consisting of n rounds.
- In phase $i \geq 0$, if there is a processor with id i, it is elected as a leader and the algorithm terminates.
- Phase i includes rounds ni+1, ni+2, …, ni+n.
- At the beginning of phase i, if a process has id i, and it has not terminated yet, the process sends a message around the ring and terminates as a leader.
- If the process does not have id i, and it receives a message in phase i, it forwards the message and terminates as the non-leader.

Bibliography

These slides are based on material that appears in the following book:

- H. Attiya & J. Welch, Distributed Computing: Fundamentals, Simulations and Advanced Topics, Morgan Kaufmann, 1998 (Chapter 3)