Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

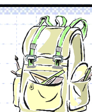
Dynamic Programming: 0/1 Knapsack

Civil War Knapsack. U.S. government image. Vicksburg National Military Park. Public domain.

The 0/1 Knapsack Problem

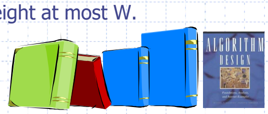
- Given: A set S of n items, with each item i having
  - w_i - a positive weight
  - b_i - a positive benefit
- Goal: Choose items with maximum total benefit but with weight at most W.
- If we are **not** allowed to take fractional amounts, then this is the **0/1 knapsack problem**.
  - In this case, we let T denote the set of items we take
  - Objective: maximize $\sum_{i \in T} b_i$
  - Constraint: $\sum_{i \in T} w_i \le W$

Example



- Given: A set S of n items, with each item i having
  - b_i - a positive "benefit"
  - w_i - a positive "weight"
- Goal: Choose items with maximum total benefit but with weight at most W.

Items:



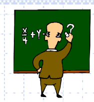
	1	2	3	4	5
Weight:	4 in	2 in	2 in	6 in	2 in
Benefit:	$20	$3	$6	$25	$80

"knapsack"

box of width 9 in

Solution:
- item 5 ($80, 2 in)
- item 3 ($6, 2in)
- item 1 ($20, 4in)

The General Dynamic Programming Technique



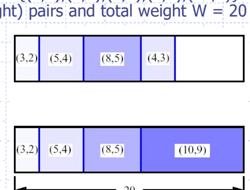
- Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
  - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
  - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).

A 0/1 Knapsack Algorithm, First Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k] = best selection from S_k.
- Problem: does not have subproblem optimality:
  - Consider set S={(3,2),(5,4),(8,5),(4,3),(10,9)} of (benefit, weight) pairs and total weight W = 20

Best for S_4:

Best for S_5:



A 0/1 Knapsack Algorithm, Second (Better) Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k,w] to be the best selection from S_k with weight at most w
- Good news: this does have subproblem optimality.

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- I.e., the best subset of S_k with weight at most w is either
  - the best subset of S_{k-1} with weight at most w or
  - the best subset of S_{k-1} with weight at most $w-w_k$ plus item k

0/1 Knapsack Algorithm

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\, B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- Recall the definition of B[k,w]
- Since B[k,w] is defined in terms of B[k–1,*], we can use two arrays of instead of a matrix
- Running time: O(nW).
- Not a polynomial-time algorithm since W may be large
- This is a pseudo-polynomial time algorithm

```
Algorithm 01Knapsack(S, W):
    Input: set S of n items with benefit b_i
           and weight w_i; maximum weight W
    Output: benefit of best subset of S with
           weight at most W
    let A and B be arrays of length W + 1
    for w ← 0 to W do
        B[w] ← 0
    for k ← 1 to n do
        copy array B into array A
        for w ← w_k to W do
            if A[w−w_k] + b_k > A[w] then
                B[w] ← A[w−w_k] + b_k
    return B[W]
```

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