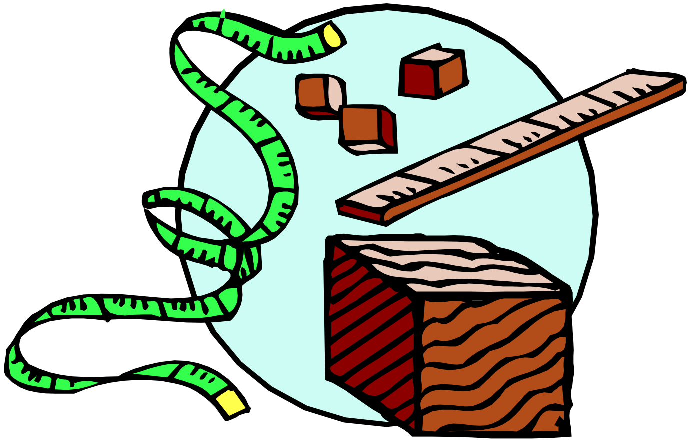
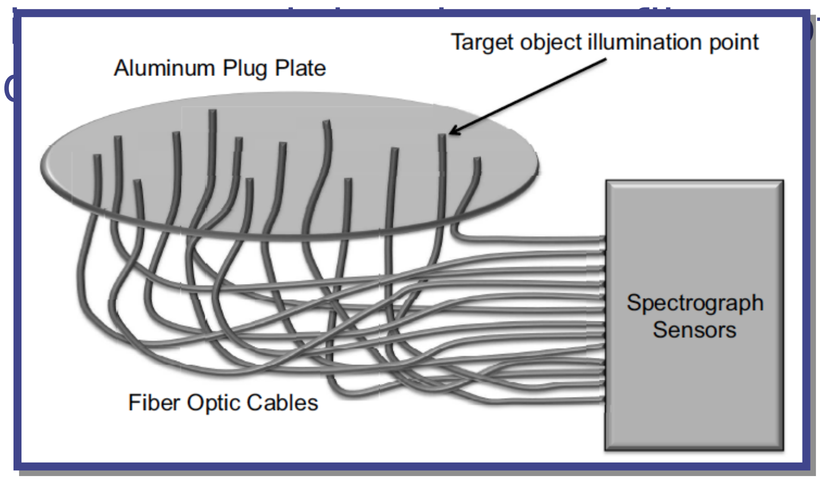
Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Approximation Algorithms



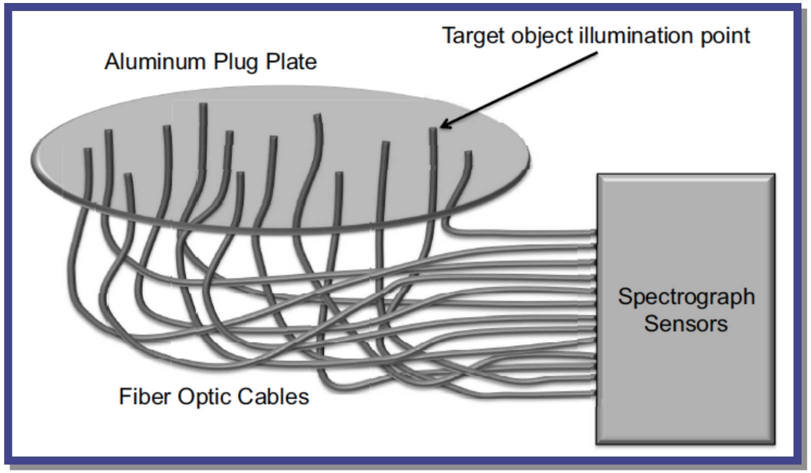
Applications

- One of the most time-consuming parts of astronomy involves collecting the light from the galaxy or star over a given period of time.
- To do this with a telescope, a large aluminum disk the size of the diameter of the telescope is used.
- This disk is placed in the focal plane of the telescope, so that the light from each stellar objects in an observation falls in a specific spot on the disk.
- The astronomers use robotic drilling equipment to drill a hole in each spot of … tic cable into each such hole and …



Application to TSP

- Drilling the holes in the fastest way is an instance of the **traveling salesperson problem (TSP)**.



- According to this formulation of TSP, each of the hole locations is a "city" and the time it takes to move a robot drill from one hole to another corresponds to the distance between the "citie" for these two holes.
- Thus, a minimum-distance tour of the cities that starts and ends at the resting position for the robot drill is one that will drill the holes the fastest.
- Unfortunately, TSP is NP-complete.
- So it would be ideal if we could at least approximate this problem.

Application to Set Cover

- Another optimization problem is to minimize the number of observations needed in order to collect the spectra of all the stellar objects of interest.
- In this case, we want to cover the map of objects with the minimum number of disks having the same diameter as the telescope.
- This optimization problem is an instance of the **set cover problem**.
- Each of the distinct sets of objects that can be included in a single observation is given as an input set and the optimization problem is to minimize the number of sets whose union includes all the objects of interest.
- This problem is also NP-complete, but it is a problem for which an approximation to the optimum might be sufficient.

Set Cover Example

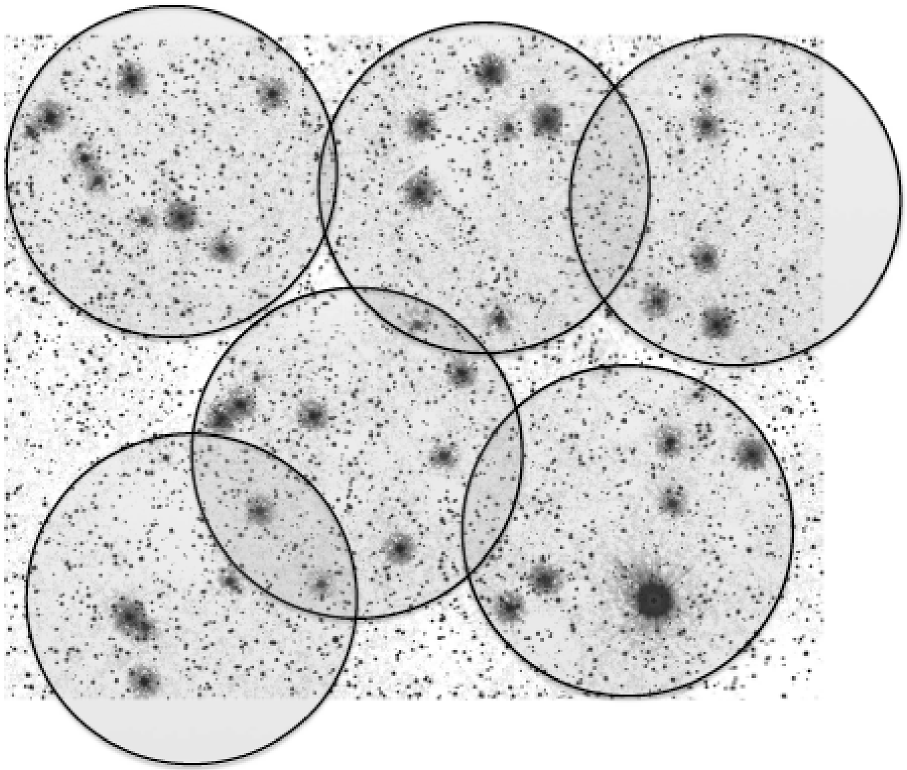
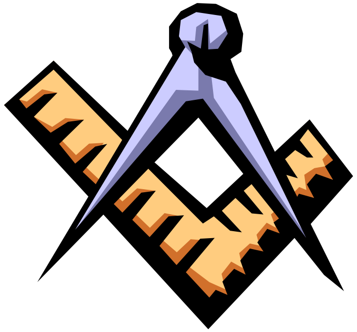


Figure 18.2: An example disk cover for a set of significant stellar objects (smaller objects are not included). Background image is from Omega Centauri, 2009. U.S. government image. Credit: NASA, ESA, and the Hubble SM4 ERO team.

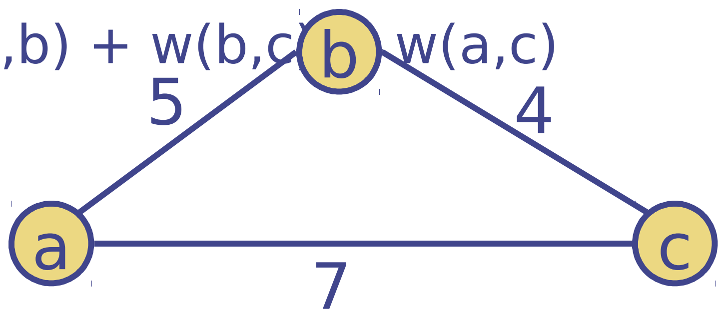
Approximation Ratios



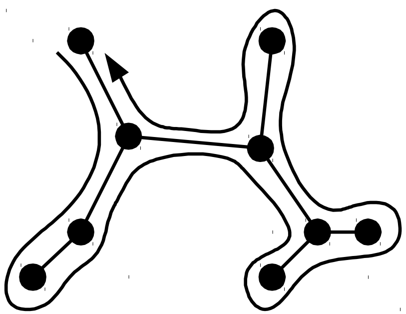
- Optimization Problems
  - We have some problem instance x that has many feasible "solutions".
  - We are trying to minimize (or maximize) some cost function c(S) for a "solution" S to x. For example,
    - Finding a minimum spanning tree of a graph
    - Finding a smallest vertex cover of a graph
    - Finding a smallest traveling salesperson tour in a graph
- An approximation produces a solution T
  - T is a **k-approximation** to the optimal solution OPT if $c(T)/c(OPT) \leq k$ (assuming a min. prob.; a maximization approximation would be the

Traveling Salesperson Problem

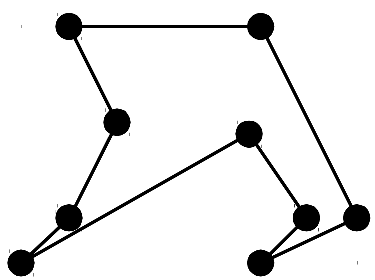
- OPT-TSP: Given a complete, weighted graph, find a cycle of minimum cost that visits each vertex.
  - OPT-TSP is NP-hard
  - Special case: edge weights satisfy the triangle inequality (which is common in many applications):
    - w(a,b) + w(b,c) ≥ w(a,c)



A 2-Approximation for TSP Special Case



Euler tour *P* of MST *M*



Output tour *T*

Algorithm *TSPApprox*(*G*)

  Input weighted complete graph ***G*,**
    satisfying the triangle inequality

  Output a TSP tour ***T*** for ***G***

  M ← a minimum spanning tree for ***G***

  P ← an Euler tour traversal of ***M*,**
    starting at some vertex ***s***

  T ← empty list

  for each vertex ***v*** in ***P*** (in traversal order)

    if this is ***v***'s first appearance in ***P*** **then**

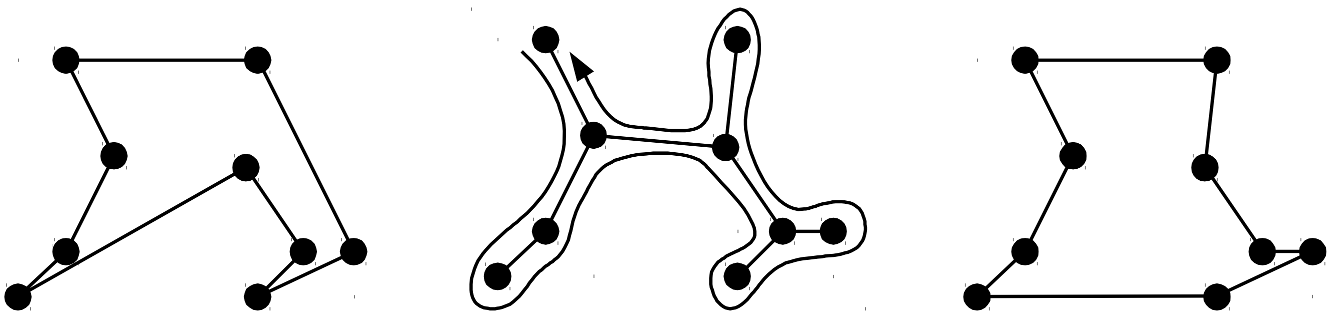
      T*.*insertLast(***v***)

  T*.*insertLast(***s***)

  return ***T***

A 2-Approximation for TSP Special Case - Proof

- The optimal tour is a spanning tour; hence |M|≤|OPT|.
- The Euler tour P visits each edge of M twice; hence |P|=2|M|
- Each time we shortcut a vertex in the Euler Tour we will not increase the total length, by the triangle inequality (w(a,b) + w(b,c) ≥ w(a,c)); hence, |T|≤|P|.
- Therefore, |T|≤|P|=2|M|≤2|OPT|



Output tour *T* (at most the cost of *P*)

Euler tour *P* of MST *M* (twice the cost of *M*)

Optimal tour *OPT* (at least the cost of MST *M*)

The Christofides Algorithm

1. Construct a minimum spanning tree, M, for G.
2. Let W be the set of vertices of G that have odd degree in M and let H be the subgraph of G induced by the vertices in W. That is, H is the graph that has W as its vertices and all the edges from G that join such vertices. By a simple argument, we can show that the number of vertices in W is even (see Exercise R-18.12). Compute a minimum-cost perfect matching, P, in H.
3. Combine the graphs M and P to create a graph, G', but don't combine parallel edges into single edges. That is, if an edge e is in both M and P, then we create two copies of e in the combined graph, G'.
4. Create an Eulerian circuit, C, in G', which visits each edge exactly once (unlike in the 2-approximation algorithm, here the edges of G' are undirected).
5. Convert C into a tour, T, by skipping over previously visited vertices.

e running time is dominated by Step 2, which takes $O(n^3)$ tin

Example and Start of Analysis

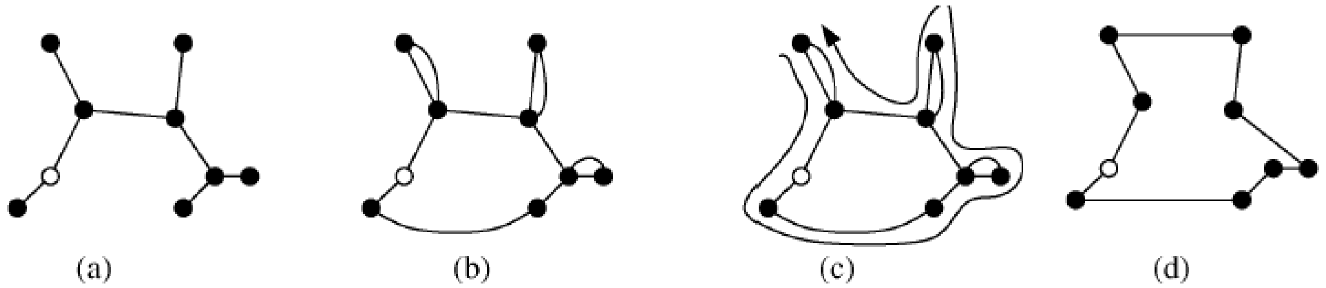


Figure 18.5: Illustrating the Christofides approximation algorithm: (a) a minimum spanning tree, M, for G; (b) a minimum-cost perfect matching P on the vertices in W (the vertices in W are shown solid and the edges in P are shown as curved arcs); (c) an Eulerian circuit, C, of G'; (d) the approximate TSP tour, T.

- To begin our analysis of the Christofides approximation algorithm, let S be an optimal solution to this instance of METRIC-TSP and let T be the tour that is produced by the Christofides approximation algorithm.
- Because S includes a spanning tree and M is a minimum spanning tree in G, c(M) ≤ c(S).

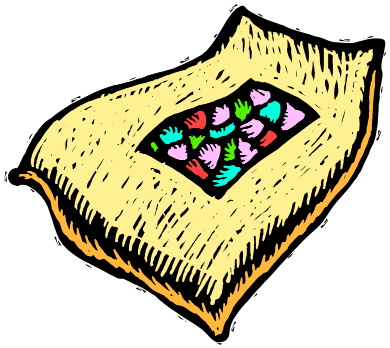
Analysis, continued

- In addition, let R denote a solution to the traveling salesperson problem on H.
- Since the edges in G (and, hence, H) satisfy the triangle inequality, and all the edges of H are also in G, $c(R) \leq c(S)$.
- That is, visiting more vertices than in the tour R cannot reduce its total cost.
- Consider now the cost of a perfect matching, P, of H, and how it relates to R, an optimal traveling salesperson tour of H. Number the edges of R, and ignore the last edge (which returns to the start vertex).
- Note that the costs of the set of odd-numbered edges and the set of even-numbered edges in R sum to c(R); hence, one of these two sets has total cost at most half of that of R, that is, cost at most $c(R)/2$.

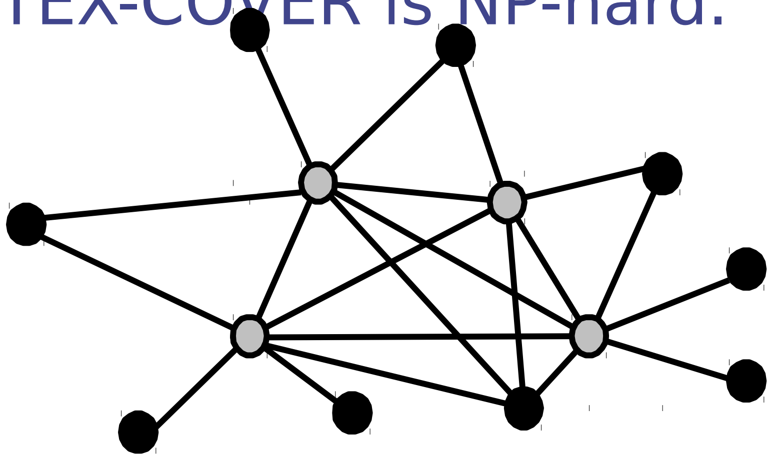
Analysis, completed

- The set of odd-numbered edges and the set of even-numbered edges in R are both perfect matchings; hence, the cost of P, a minimum-weight perfect matching on the edges of H, will be at most the smaller of these two. That is, $c(P) \le c(R)/2$.
- Therefore, $c(M) + c(P) \le c(S) + c(R)/2 \le 3c(S)/2$.
- Since the edges in G satisfy the triangle inequality, we can only improve the cost of a tour by making shortcuts that avoid previously visited vertices. Thus, $c(T) \le c(M) + c(P)$, which implies that $c(T) \le 3c(S)/2$.
- In other words, the Christofides approximation algorithm gives us a **(3/2)-approximation algorithm** for the METRIC-TSP optimization problem that runs in polynomial time.

Vertex Cover



- A **vertex cover** of graph G=(V,E) is a subset W of V, such that, for every (a,b) in E, a is in W or b is in W.
- OPT-VERTEX-COVER: Given an graph G, find a vertex cover of G with smallest size.
- OPT-VERTEX-COVER is NP-hard.



A 2-Approximation for Vertex Cover

- Every chosen edge e has both ends in C
- But e must be covered by an optimal cover; hence, one end of e must be in OPT
- Thus, there is at most twice as many vertices in C as in OPT.
- That is, C is a 2-approx. of OPT
- Running time: ...

Algorithm VertexCoverApprox(G):
    Input: A graph G
    Output: A small vertex cover C for G
    $C \leftarrow \emptyset$
    while G still has edges **do**
        select an edge $e = (v, w)$ of G
        add vertices v and w to C
        for each edge f incident to v or w **do**
            remove f from G
    return C

Set Cover (Greedy Algorithm)

- OPT-SET-COVER: Given a collection of m sets, find the smallest number of them whose union is the same as the whole collection of m sets?
  - OPT-SET-COVER is NP-hard
- Greedy approach produces an O(log n)-approximation algorithm.

Algorithm SetCoverApprox(S):
    Input: A collection S of sets $S_1, S_2, \ldots, S_m$ whose union is U
    Output: A small set cover C for S

```
C ← ∅        // The set cover built so far
E ← ∅        // The elements from U currently covered by C
while E ≠ U do
    select a set S_i that has the maximum number of uncovered elements
    add S_i to C
    E ← E ∪ S_i
Return C.
```

Greedy Set Cover Analysis

- Consider the moment in our algorithm when a set S_j is added to C, and let k be the number of previously uncovered elements in S_j.
- We pay a total charge of 1 to add this set to C, so we charge each previously uncovered element i of S_j a charge of c(i) = 1/k.
- Thus, the total size of our cover is equal to the total charges made.
- To prove an approximation bound, we will consider the charges made to the elements in each subset S_j that belongs to an optimal cover, C ′. So, suppose that S_j belongs to C ′.
- Let us write $S_j = \{x_1, x_2, . . . , x_{nj}\}$ so that S_j's elements are listed in the order in which they are covered by our algorithm.

Greedy Set Cover Analysis, cont.

Now, consider the iteration in which x_1 is first covered. At that moment, S_j has not yet been selected; hence, whichever set is selected must have at least n_j uncovered elements. Thus, x_1 is charged at most $1/n_j$. So let us consider, then, the moment our algorithm charges an element x_l of S_j. In the worst case, we will have not yet chosen S_j (indeed, our algorithm may never choose this S_j). Whichever set is chosen in this iteration has, in the worst case, at least $n_j - l + 1$ uncovered elements; hence, x_l is charged at most $1/(n_j - l + 1)$. Therefore, the total amount charged to all the elements of S_j is at most

$$\sum_{l=1}^{n_j} \frac{1}{n_l - l + 1} = \sum_{l=1}^{n_j} \frac{1}{l},$$

which is the familiar ***harmonic number***, H_{n_i}. It is well known (for example, see the Appendix) that H_{n_j} is $O(\log n_j)$. Let $c(S_j)$ denote the total charges given to all the elements of a set S_j that belongs to the optimal cover C'. Our charging scheme implies that $c(S_j)$ is $O(\log n_j)$. Thus, summing over the sets of C', we obtain

$$\begin{aligned} \sum_{S_j \in C'} c(S_j) &\leq \sum_{S_j \in C'} b \log n_j \\ &\leq b|C'| \log n, \end{aligned}$$

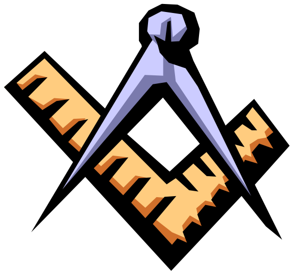
for some constant $b \geq 1$. But, since C' is a set cover,

$$\sum_{i \in U} c(i) \leq \sum_{S_j \in C'} c(S_j).$$

Therefore,

$$|C| \leq b|C'| \log n.$$

Polynomial-Time Approximation Schemes



- A problem L has a **polynomial-time approximation scheme (PTAS)** if it has a polynomial-time (1+ε)-approximation algorithm, for any fixed $\varepsilon > 0$ (this value can appear in the running time).
- 0/1 Knapsack has a PTAS, with a running time that is $O(n^3/\varepsilon)$.