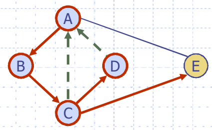
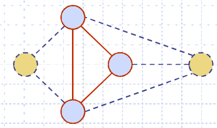
Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

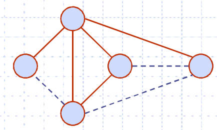
Depth-First Search



Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G


Subgraph

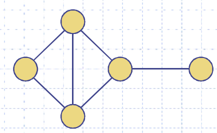

Spanning subgraph

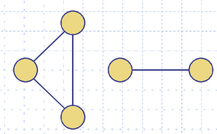
Application: Web Crawlers

- A fundamental kind of algorithmic operation that we might wish to perform on a graph is **traversing the edges and the vertices** of that graph.
- A **traversal** is a systematic procedure for exploring a graph by examining all of its vertices and edges.
- For example, a **web crawler**, which is the data collecting part of a search engine, must explore a graph of hypertext documents by examining its vertices, which are the documents, and its edges, which are the hyperlinks between documents.
- A traversal is efficient if it visits all the vertices and edges in linear time.

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G

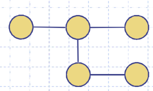

Connected graph

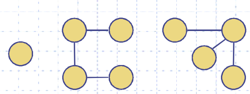

Non connected graph with two connected components

Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

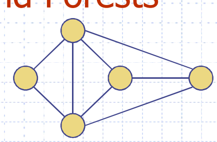
  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees

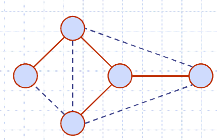

Tree


Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest


Graph


Spanning tree

Depth-First Search

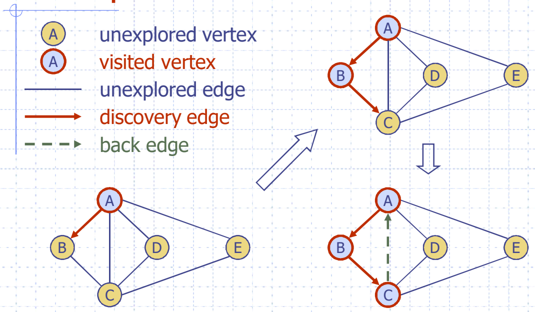
- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees

DFS Algorithm from a Vertex

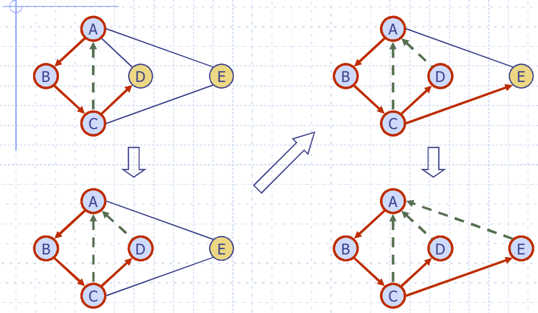
Algorithm DFS(G, v):
Input: A graph G and a vertex v in G
Output: A labeling of the edges in the connected component of v as discovery edges and back edges, and the vertices in the connected component of v as explored

Label v as explored
for each edge, e, that is incident to v in G **do**
  if e is unexplored **then**
    Let w be the end vertex of e opposite from v
    if w is unexplored **then**
      Label e as a discovery edge
      DFS(G, w)
    else
      Label e as a back edge

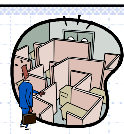
Example



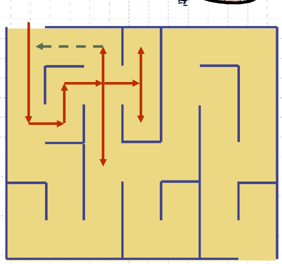
Example (cont.)



DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



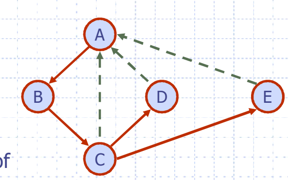
Properties of DFS

Property 1

$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v

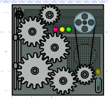


The General DFS Algorithm

- Perform a DFS from each unexplored vertex:

```
Algorithm DFS(G):
    Input: A graph G
    Output: A labeling of the vertices in each connected component of G as explored
  Initially label each vertex in v as unexplored
  for each vertex, v, in G do
      if v is unexplored then
          DFS(G, v)
```


Analysis of DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

Path Finding (not in book)

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop(e)
      else
        setLabel(e, BACK)
  S.pop(v)
```


Cycle Finding (not in book)

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      S.push(e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        pathDFS(G, w, z)
        S.pop(e)
      else
        T ← new empty stack
        repeat
          o ← S.pop()
          T.push(o)
        until o = w
        return T.elements()
  S.pop(v)
```