

Exercise 1

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HY-370

Let S_1 be a causal and stable LTI system with impulse response $h_1[n]$ and frequency response $H_1(e^{j\omega})$, described by:

$$y[n] - y[n-1] + \frac{1}{4} y[n-2] = x[n], \quad (i).$$

I) If LTI system S_2 has frequency response $H_2(e^{j\omega}) = H_1(-e^{j\omega})$, is it highpass, lowpass, or bandpass?

• $\{ (i) \} \Rightarrow Y(z) \left(1 - z^{-1} + \frac{1}{4} z^{-2} \right) = X(z) \Rightarrow$

$$\Rightarrow H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{4} z^{-2}} \quad (=)$$

$$(=) \quad H_1(z) = \frac{1}{\left(1 - \frac{1}{2} z^{-1} \right)^2}$$

• $H_2(e^{j\omega}) = H_1(-e^{j\omega}) \Rightarrow h_2[n] = -h_1[n] \Rightarrow \underline{H_2(z) = H_1(-z)}$

Hence: $H_2(z) = \frac{1}{\left(1 + \frac{1}{2} z^{-1} \right)^2}$

→ Double pole at $z^{-1} = -2 \quad (=)$

$$(=) \quad 2 = -\frac{1}{z} \quad (=) \quad \boxed{z = \frac{1}{2} e^{j\pi}}$$

- This means that frequencies around $\omega = \pi$ are boosted, making $H_2(z)$ a highpass filter (poles make the 2 transform go towards infinity there).

II) Let S_3 be a causal LTI system with $H_3(e^{j\omega}) H_1(e^{j\omega}) = 1$.

Is it minimum phase? Is it among the four types of linear phase systems that you know of?

$$\bullet H_3(e^{j\omega}) H_1(e^{j\omega}) = 1 \Rightarrow h_3[n] * h_1[n] = \delta[n] \Rightarrow$$

$$\Rightarrow H_3(z) H_1(z) = 1 \Rightarrow \boxed{H_3(z) = \frac{1}{H_1(z)}} \text{, hence:}$$

$$\bullet \boxed{H_3(z) = \left(1 - \frac{1}{2}z^{-1}\right)^2} \rightarrow \text{2-nd rank zero at } \boxed{z = \frac{1}{2}e^{j0}}$$

- Therefore, it hinders frequencies around $\omega = 0$, making it a highpass filter.
- Its zeros are in the unit circle, so it is also minimum phase.
- Its zeros are not in conjugate mutual pairs, making it not linear phase.

• We can also see the latter from the time domain:

$$Z^{-1} \{ H_3(z) \} = h_3[n] = \delta[n] - \delta[n-1] + \frac{1}{4} \delta[n-2],$$

where no characteristic symmetry of FIR linear phase system is observed.

Exercise 2

• We have 3 causal and real LTI systems with transfer functions $H_1(z)$, $H_2(z)$, $H_3(z)$ respectively. Find as much information about them as you can, regarding:

(a) Their poles and zeroes.

(b) The duration of their impulse response.

I) It is given that: $H_1(z)$ has a pole at $z = 0.9 e^{j\pi/3}$, and for input $x[n] = u[n]$, it holds that $\lim_{n \rightarrow +\infty} y[n] = 0$, where $y[n]$ is the output of the system.

(a) • The system is real, therefore its poles and zeroes come in conjugate pairs, or, its poles and zeroes are real.

• So, it must have another pole at $z^* = 0.9 e^{-j\pi/3}$.

• Causal means that it has no poles at infinity.

• We know that: $x[n] = u[n] \Rightarrow X(z) = \frac{1}{1-z^{-1}}, |z| > 1$.

The output in this case at the z domain will have the form:

$$Y(z) = H_1(z) X(z) = \frac{H_1(z)}{1-z^{-1}}, \text{ and since we know that}$$

for this input, it holds that $\lim_{n \rightarrow +\infty} y[n] = 0$, this indicates

that the pole at $z=1$ must be cancelled with a zero at $z=1$.

• Another way to think about it, is that if we perform partial fraction decomposition for $\frac{H_1(z)}{1-z^{-1}}$, then we will

get a term $A \cdot \frac{1}{1-z^{-1}}$. In the time domain, this term

translates to $z^{-1} \sum A / (1-z^{-1}) = Au[n]$, but this term

will not fade to 0 as $n \rightarrow +\infty$ (it will remain constant = A). Hence,

to satisfy $\lim_{n \rightarrow +\infty} y[n] = 0$, this term must be cancelled, so

$H_1(z)$ must have $1-z^{-1}$ in the numerator, i.e., a zero at $z=1$.

• Therefore, in total, we have:

$$H_1(z) = \frac{1-z^{-1}}{(1-0.9e^{j\pi/3}z^{-1})(1-0.9e^{-j\pi/3}z^{-1})} H_0(z),$$

where $H_0(z)$ keeps the rest of the information that we do not know about this system.

(b) Since there is at least one pole for $H_1(z)$ (that is not cancelled out), then its impulse response must be of infinite duration. A term of the form $\frac{A}{1-az^{-1}}$, $0 < |a| < 1$, for example, will transform into $Aa^n u[n]$, which is non-zero $\forall n \geq 0$, thus continuing indefinitely.

II) $H_2(z)$ has a zero at $z = 0.8e^{j\pi/4}$, it is linear phase with group delay $\text{grd} \left\{ \angle H_2(e^{j\omega}) \right\} = -2.5$, and $|H_2(e^{j0})| = 0$.

(a) Again, real means that zeroes come in mutual conjugate pairs $(z, z^*, \frac{1}{z}, \frac{1}{z^*})$, so it will have zeroes at $(0.8e^{j\pi/4}, 0.8e^{-j\pi/4}, 1.25e^{j\pi/4}, 1.25e^{-j\pi/4})$.

• Causal + linear phase means that it has to be one of the four types that we know from theory. We have:

$$\text{grd} \left\{ \angle H_2(e^{j\omega}) \right\} = -2.5 = -\frac{5}{2}, \text{ so:}$$

$$\angle H_2(e^{j\omega}) = -\omega \frac{5}{2} + C, \text{ where } C \text{ is a constant.}$$

{ group delay = negative derivative of the phase. }

• Thus, $M = 5$ in our case, which is an odd number. So, this is an either type II or type IV linear phase system. Therefore, all of its poles are located at the origin $z = 0$.

• $|H_2(e^{j\omega})| = 0$ means that at $\omega = 0$, or $z = 1$ there is a zero. That makes it a type IV linear phase system.

• Since $M = 5$, there is a total of 5 zeroes for $H_2(z)$ and we have found all 5 of them. In total, we can say that:

$$H_2(z) = A(1 - z^{-1})(1 - z_1)(1 - z_1^*)(1 - 1/2)(1 - 1/2^*),$$

where $z_1 = 0.8e^{j\pi/4}$, $A \in \mathbb{R}$, almost fully describing this system.

(b) From theory, the duration of all type I, II, III, IV linear phase systems' $h[n]$ is $M+1$ samples. Hence, $h_2[n]$ has 6 delta functions, i.e., its duration is 6 samples long.

III) $H_3(z)$ has a pole at $z = 0.8e^{j\pi/4}$, and $|H_3(e^{j\omega})| = 1, \forall \omega$.

(a) $|H_3(e^{j\omega})| = 1, \forall \omega \Rightarrow$ this is an all-pass system.

Therefore, its zeroes and poles come in conjugate mutual pairs:

Pole at $z = 0.8e^{j\pi/4} \Rightarrow$ pole at $z^* = 0.8e^{-j\pi/4} \Rightarrow$

zeros at $1/2 = 1.25e^{j\pi/4}$, $1/2^* = 1.25e^{-j\pi/4}$.

Hence, we collected that:

$$H_3(z) = \frac{(z^{-1} - 0.8e^{j\pi/4})(z^{-1} - 0.8e^{-j\pi/4})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} H_{ap}(z),$$

where $H_{ap}(z)$ holds the rest of the information that we do not know about $H_3(z)$, and it is likewise also necessarily all-pass.

(b) Similarly, when we have a system with a pole that does not cancel out, then its impulse response is of infinite duration (IIR). Remember, a pole represents a term in the transfer function that causes a system's response to be dominated by an exponential decay or growth, depending on its position in the z -plane. Only a zero in the same position would nullify the pole's effect. Important: This is true, of course, for poles not at zero or infinity. If a system's poles are all located at zero or infinity, then it is FIR.

Exercise 3

Two LTI systems, $H_1(e^{j\omega})$, and $H_2(e^{j\omega})$, have generalized linear phase. Which of the following systems are also generalized linear phase?

(a) $G_1(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$

- The phase of this system is given by:

$$\angle G_1(e^{j\omega}) = \tan^{-1} \left(\frac{\text{Im} \left\{ \sum H_1(e^{j\omega}) + H_2(e^{j\omega}) \right\}}{\text{Re} \left\{ \sum H_1(e^{j\omega}) + H_2(e^{j\omega}) \right\}} \right),$$

which is not always linear. We can give an example:

$$h_1[n] = \delta[n] + 2\delta[n-1], \quad h_2[n] = 2\delta[n] - 2\delta[n-1].$$

It will be: $g_1[n] = h_1[n] + h_2[n] = 3\delta[n] - \delta[n-1]$, so:

$$G_1(e^{j\omega}) = \mathcal{F} \left\{ 3\delta[n] - \delta[n-1] \right\} = 3 - e^{-j\omega} = 3 - \cos(\omega) + j\sin(\omega).$$

Hence: $\angle G_1(e^{j\omega}) = \tan^{-1} \left(\frac{\sin(\omega)}{3 - \cos(\omega)} \right)$, not a linear phase,

but, $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are linear phase systems.

(b) $G_2(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$

- The phase will be $\angle G_2(e^{j\omega}) = \angle H_1(e^{j\omega}) + \angle H_2(e^{j\omega})$. Summing two lines always gives another line, so this system is always a generalized linear phase system.

(c) $G_3(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_1(e^{j\theta}) H_2(e^{j(\omega-\theta)}) d\theta.$

- A counter-example is an easy way to prove that this system is not necessarily generalized linear phase:

Remember that: $\mathcal{F}\{x[n]y[n]\} = X(e^{j\omega}) * Y(e^{j\omega}) =$
 $= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$, by definition.

Therefore, for this system: $g_3[n] = h_1[n]h_2[n]$.

• As for the counter-example, we take two linear phase systems:

$$h_1[n] = \delta[n] + \delta[n-1], \quad h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

$$\text{So: } g_3[n] = (\delta[n] + \delta[n-1]) (\delta[n] + 2\delta[n-1] + \delta[n-2])$$

$$x[n-n_1] \delta[n-n_2] = x[n_2-n_1] \delta[n-n_2]$$

$$= \delta[0] \delta[n] + \delta[1] \cdot 2\delta[n-1] + \delta[2] \delta[n-2] +$$

$$+ \delta[-1] \delta[n] + \delta[0] 2\delta[n-1] + \delta[1] \delta[n-2]$$

$$= \delta[n] + 2\delta[n-1] = g_3[n].$$

• Thus: $G_3(e^{j\omega}) = \mathcal{F}\{\delta[n] + 2\delta[n-1]\} = 1 + 2e^{-j\omega} = 1 + 2\cos(\omega) - 2j\sin(\omega)$,

giving a non-linear phase: $\angle G_3(e^{j\omega}) = \tan^{-1}\left(\frac{2\sin(\omega)}{1+2\cos(\omega)}\right)$.

$$\delta[n-n_0] \xrightarrow{\mathcal{F}} e^{-j\omega n_0} = \cos(\omega \cdot n_0) - j\sin(\omega \cdot n_0), \text{ from:}$$

$$e^{jx} = \cos(x) + j\sin(x).$$

Exercise 4 (From 2016 HY-370 Final exam)

The Fibonacci sequence $(0, 1, 1, 2, 3, 5, 8, \dots)$ can be modelled as the impulse response of the following system:

$$y[n] - y[n-1] - y[n-2] = x[n-1], \quad (i)$$

Calculate this impulse response $h[n]$.

• $\mathcal{Z}\{(i)\} \Leftrightarrow Y(z) (1 - z^{-1} - z^{-2}) = X(z) z^{-1} \quad (=)$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}, \quad (ii)}$$

with poles at $1 - z^{-1} - z^{-2} = 0 \quad (=) \quad z^2 - z - 1 = 0$

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot (-1) = 5, \quad \boxed{z_{1,2} = \frac{1 \pm \sqrt{5}}{2}, \quad (iii)}$$

• Partial Fraction Decomposition on (ii):

$$\frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{A}{1 - z_1 z^{-1}} + \frac{B}{1 - z_2 z^{-1}} \quad (=) \quad z^{-1} = A(1 - z_2 z^{-1}) + B(1 - z_1 z^{-1}).$$

• For $z^{-1} = z_2^{-1}$: $\frac{1}{z_2} = B \left(1 - \frac{z_1}{z_2}\right) \quad (=) \quad B = \frac{1}{z_2 - z_1} \stackrel{(iii)}{\Rightarrow} \underline{B = -1/\sqrt{5}}$

• For $z^{-1} = z_1^{-1}$: $\frac{1}{z_1} = A \left(1 - \frac{z_2}{z_1}\right) \quad (=) \quad A = \frac{1}{z_1 - z_2} \stackrel{(iii)}{\Rightarrow} \underline{A = 1/\sqrt{5}}$

So, in total:

$$H(z) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1+\sqrt{5}}{2} z^{-1}} \right) - \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1-\sqrt{5}}{2} z^{-1}} \right), \quad (iv)$$

- But, how can we do the inverse without knowing the ROC?
- Observe that the Fibonacci sequence increases indefinitely, so this system is not stable. \Rightarrow unit circle not in ROC.
- From (i), we can see that there is no dependence on future values of the input in order to get the output for this system. Hence, this system is causal. $\Rightarrow |z| > a, a \in \mathbb{R}$.
- From the above, the ROC is $|z| > \frac{1+\sqrt{5}}{2} \approx 1.618\dots$, so now we can use the corresponding known pairs to get to the time domain:

$$z^{-1} \{ (iv) \} \Leftrightarrow h[n] = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n u[n] - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n u[n].$$

Exercise 5 (From HY-370 2016 final exam)

The real part of the Fourier transform of a real and causal signal $x[n]$ is $X_R(e^{j\omega}) = 2 - 4\cos(3\omega)$. Find $X(e^{j0})$ and $X(e^{j\pi})$.

• From our theory (see table 13.4 page 721), we know that for a real signal $x[n]$, the real part of its Fourier transform corresponds to its even part, i.e., $x_e[n] \xleftrightarrow{F} X_R(e^{j\omega})$.

This is called a Hermitian property for $X(e^{j\omega})$.

• So we can find this even part:

$$X_R(e^{j\omega}) = 2 - 4\cos(3\omega) = 2 - 2e^{j3\omega} - 2e^{-j3\omega} \quad \leftarrow F^{-1}$$

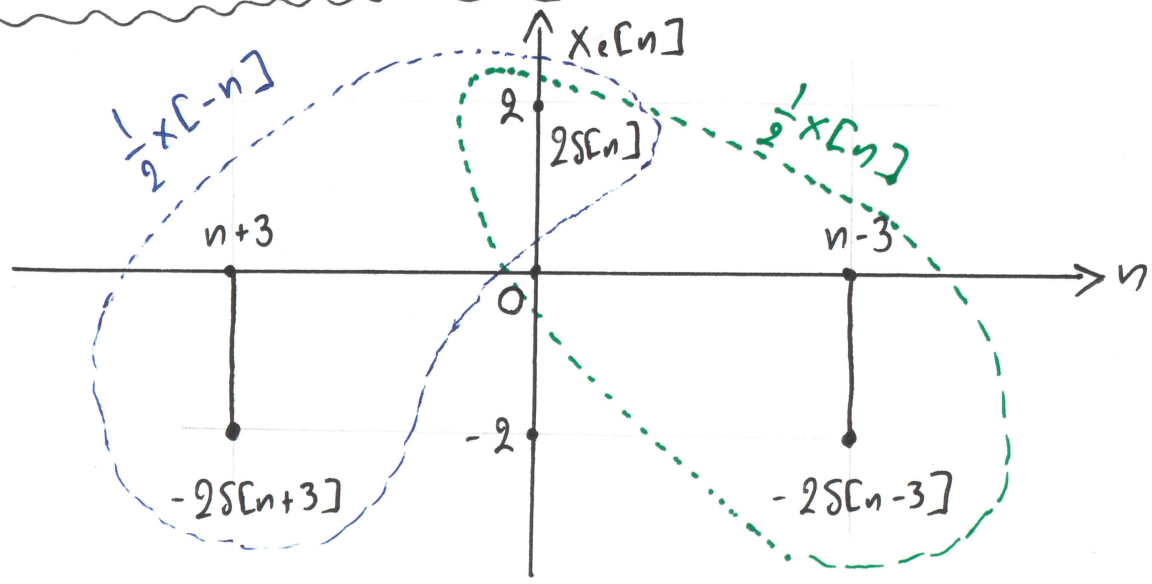
$$x_e[n] = 2\delta[n] - 2\delta[n+3] - 2\delta[n-3]$$

• Even part means that: $x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n]$, or:

$$x[n] = 2x_e[n] - x[-n]$$

For the odd part, it would be $x_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n]$.

Notice that $x_e[n] + x_o[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n] + \frac{1}{2}x[n] - \frac{1}{2}x[-n] = x[n]$



• Hence, we can see (also from the plot) that our signal:
 $x[n] = 2\delta[n] - 4\delta[n-3]$, which is causal as we know from
 the exercise.

• We can also see from the plot that $x[-n] = -2\delta[n+3] + 2\delta[n]$.
 As a sanity check, we can plug it in $x[n] = 2x_e[n] - x[-n]$
 to see if we get the same $x[n]$ algebraically like so:

$$x[n] = 2x_e[n] - x[-n] =$$

$$= 2(2\delta[n] - 2\delta[n+3] - 2\delta[n-3]) - (-2\delta[n+3] + 2\delta[n])$$

∅ system is causal, so it must be zero for all $n < 0$,

therefore, $\delta[n+3]$ is discarded:

$$= 4\delta[n] - 4\delta[n-3] - 2\delta[n] = \boxed{2\delta[n] - 4\delta[n-3] = x[n]}$$

• Now, we can evaluate what's been asked:

$$X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n] e^{j\omega n} \Big|_{\omega=0} = \sum_{n=-\infty}^{+\infty} x[n] = 2 - 4 = -2.$$

$$X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x[n] e^{j\omega n} \Big|_{\omega=\pi} = \sum_{n=-\infty}^{+\infty} x[n] (-1)^n = 2(-1)^0 - 4(-1)^3 =$$

$$= 2 + 4 = 6.$$

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