4º Φροντιστήριο

14/11/2022
HY-370

Exercise 1

- Pole - zero plot of : $x[n] = \begin{cases} a^n, & 0 \leq n \leq M-1, \ a \in \mathbb{R}, M \in \mathbb{Z}^+ \\ 0, & \text{elsewhere} \end{cases}$

Solution

- Z - Transform definition: $X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$, so: in our case:

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} = \sum_{n=0}^{M-1} (az^{-1})^n =$$

$$= \frac{1-(az^{-1})^M}{1-az^{-1}} \cdot \frac{z^M}{z^M} = \frac{z^M - a^M \cdot z^{-M+M}}{z^M - a^M z^{-M}} =$$

$$= \frac{z^M - a^M \cdot z^0}{z^M(1-az^{-1})} = \frac{z^M - a^M}{z^{M-1}(z-a)}, \ z \in \mathbb{C}.$$

↳ Zeros are found by solving the equation:

$$z^M - a^M = 0 \iff$$

$$z^M = a^M$$

↳ Since we have a polynomial of degree M, there must be a total of M roots. (careful: $z=a$ is just one root!)

↳ If we write z and a in different (exponential) forms it will help

↳ Any complex number z can be written as $z = r \cdot e^{j\vartheta}$

↳ So we can also write a^M as $\underbrace{|a|^M \cdot e^{j2\pi k}, \; k = 0, \dots, M-1}_{\text{this just changes the sign}}$

Hence:

$$z^M = a^M \iff$$

$$(re^{j\vartheta})^M = |a|^M e^{j2\pi k}, \quad k = 0, \dots, M-1 \iff$$

$$r^M \cdot e^{j\vartheta M} = |a|^M \cdot e^{j2\pi k}, \quad -\text{''}- \iff$$

$$r = |a|, \; \vartheta = 2\pi k / M, \quad -\text{''}-$$

↳ So we have M total zeros in the positions $z = |a| \cdot e^{j2\pi k/M}$, $k = 0, \dots, M-1$.

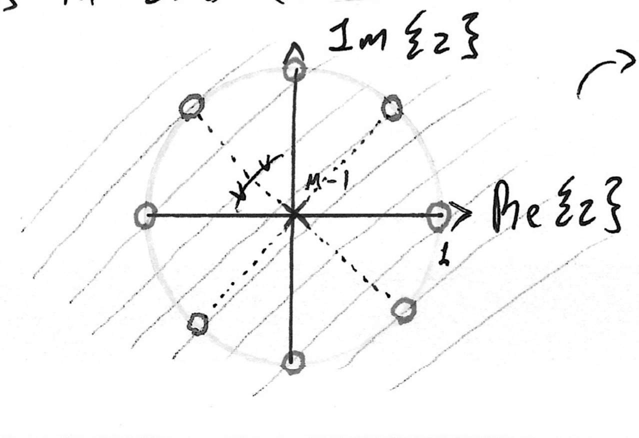
↳ The poles are in the positions:

$$z^{M-1}(z-a) = 0 \iff$$

$$z = 0, \text{ or}, z = a$$

there is both a pole and a zero at $(k=0)$ $z = a$, so they cancel each other out!

↳ So, in total we have $M-1$ zeros in $z_k = |a| e^{j2\pi k/M}$, $k = 1, \dots, M$, and $M-1$ poles in $z = 0$ ($M-1$-th order).



↳ Example with $M = 8$ and $a = 1$.

Exercise 2

Pole-zero plot of: $x[n] = (1+n)\,u[n]$

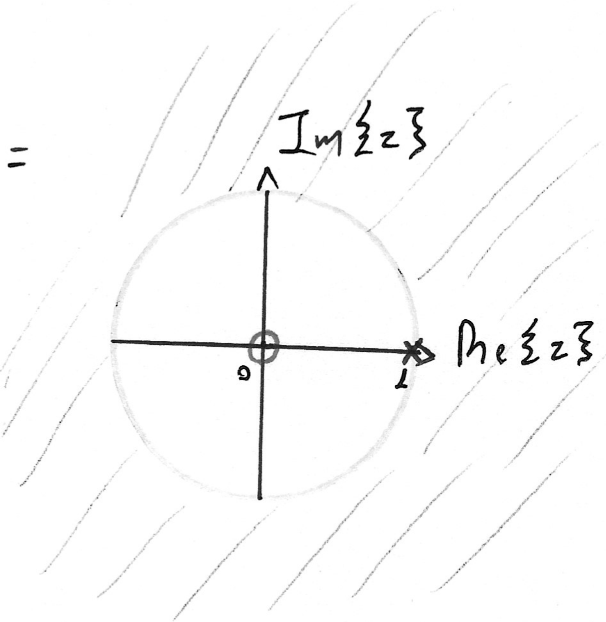
Solution

$$X(z) = \sum_{n=-\infty}^{+\infty} (1+n)\,u[n]\,z^{-n} = \sum_{n=0}^{+\infty} (1+n)\,z^{-n} =$$

$$= \sum_{n=0}^{+\infty} z^{-n} + \sum_{n=0}^{+\infty} n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^n + \sum_{n=0}^{+\infty} n\left(\frac{1}{z}\right)^n =$$

$$\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a}, \quad |a|<1, \qquad \sum_{n=0}^{+\infty} n a^n = \frac{a}{(1-a)^2}, \quad |a|<1$$

$$= \frac{1}{1-\frac{1}{z}} + \frac{\frac{1}{z}}{\left(1-\frac{1}{z}\right)^2}, \quad \left|\frac{1}{z}\right| < 1 =$$



$$= \frac{1-\frac{1}{z}+\frac{1}{z}}{\left(1-\frac{1}{z}\right)^2}, \quad |z|>1 =$$

$$= \frac{1}{\left(1-\frac{1}{z}\right)^2} \cdot \frac{z^2}{z^2}, \quad |z|>1 =$$

$$= \frac{z^2}{(z-1)^2}, \quad |z|>1 \quad \left\{ \begin{array}{l} \text{Zero at } z=0 \text{ (2-nd order)} \\ \text{Pole at } z=1 \text{ (-- '' --)} \end{array} \right.$$

Exercise 3

Pole - zero plot of: $x[n] = (a^n + a^{-n})\, u[n]$, $a \in \mathbb{R}^*$

Solution

$$X(z) = \sum_{n=-\infty}^{+\infty} (a^n + a^{-n})\, u[n]\, z^{-n} = \sum_{n=0}^{+\infty} a^n z^{-n} + \sum_{n=0}^{+\infty} a^{-n} z^{-n} =$$

$$= \sum_{n=0}^{+\infty} \left(\frac{a}{z}\right)^n + \sum_{n=0}^{+\infty} \left(\frac{1}{az}\right)^n =$$

$$\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$= \frac{1}{1 - az^{-1}} + \frac{1}{1 - (az)^{-1}}, \quad \left|\frac{a}{z}\right| < 1, \quad \left|\frac{1}{az}\right| < 1, \quad a \in \mathbb{R}^*$$

$$= \frac{1 - (az)^{-1} + 1 - az^{-1}}{(1 - az^{-1})(1 - (az)^{-1})}, \quad \left|\frac{1}{z}\right| < \frac{1}{|a|}, \quad \left|\frac{1}{z}\right| < |a|, \quad a \in \mathbb{R}^*$$

$$= \frac{2 - z^{-1}(a^{-1} + a)}{(1 - az^{-1})(1 - (az)^{-1})} \cdot \frac{z^2}{z^2}, \quad |z| > |a|, \quad |z| > |1/a|, \quad a \in \mathbb{R}^*$$

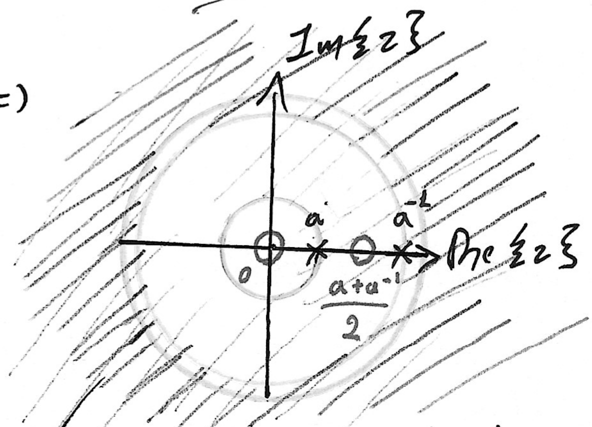
$$= \frac{2z^2 - z(a^{-1} + a)}{(z - a)(z - a^{-1})}, \quad \{|z| > |a|\} \cap \{|z| > 1/|a|\}, \quad a \in \mathbb{R}^*$$

↳ "whichever is greater"

$$= \frac{z(2z - a - a^{-1})}{(z - a)(z - a^{-1})}, \quad |z| > \max\{|a|, 1/|a|\}, \quad a \in \mathbb{R}^*$$

↳ for the ROC not to be $\emptyset$.

Example



↳ a^{-1} and a could be in the opposite places if $|a|>1$, ROC different as well.

↳ Zeros: $z(2z - a - a^{-1}) = 0 \Leftrightarrow$

$z = 0$, or $2z - a - a^{-1} = 0 \Leftrightarrow$

$z = 0$, or $z = \frac{a + a^{-1}}{2}$.

↳ Poles: $(z - a)(z - a^{-1}) = 0 \Leftrightarrow$

$z = a$, or $z = a^{-1}$.

Exercise 4

Pole-zero plot of: $x[n] = a^{|n|}$, $0 < a < 1$

Solution

$$X(z) = \sum_{n=-\infty}^{+\infty} a^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{+\infty} a^{n} z^{-n} =$$

$$= \sum_{n=1}^{+\infty} (az)^n + \sum_{n=0}^{+\infty} \left(\frac{a}{z}\right)^n =$$

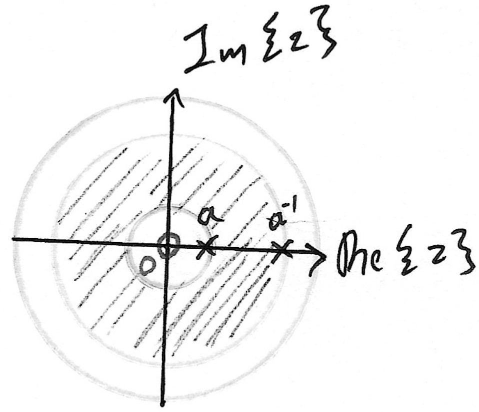
$$\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a}, \quad |a| < 1, \quad \sum_{n=N_1}^{+\infty} a^n = \frac{a^{N_1}}{1-a}, \quad |a| < 1$$

$$= \frac{az}{1-az} + \frac{1}{1-a/z}, \quad |az| < 1, \quad \left|\frac{a}{z}\right| < 1, \quad 0 < a < 1$$

$$= \frac{az}{1-az} + \frac{z}{z-a}, \quad |z| < 1/|a|, \quad \frac{1}{|z|} < \frac{1}{|a|}, \quad 0 < a < 1$$

$$= \frac{az(z-a) + z(1-az)}{(1-az)(z-a)}, \quad |z| < 1/|a|, \quad |z| > |a|, \quad 0 < a < 1$$

$$= \frac{z(1-a^2)}{(1-az)(z-a)}, \quad |a| < |z| < 1/|a|, \quad 0 < a < 1$$



↳ Poles: $(1-az)(z-a) = 0 \iff$

$1 - az = 0$, or $z - a = 0 \iff$

$z = 1/a$, or $z = a$.

↳ Zeros: $z(1-a^2) = 0 \iff$

$z = 0$, but the number of zeros should* equal the number of poles! The other zero in this case is "hidden" at $+\infty$. Observe that $\lim_{z \to +\infty} X(z) = 0$.

* always

Exercise 5

Causal LTI sysem which for this input:

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + 2^n u[-n-1]$$

produces this output:

$$y[n] = 5\left(\frac{1}{3}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n]$$

Find : i) Its transfer function $H(z)$,

ii) Its impulse response $h[n]$.

iii) Its difference equation.

i) We know that it is given by: $H(z) = Y(z) / X(z)$, (1). So:

$$X(z) = Z\left\{ \left(\frac{1}{3}\right)^n u[n] + 2^n u[-n-1] \right\} =$$

known pairs: $a^n u[n] \xrightarrow{Z} \dfrac{1}{1-az^{-1}}$, $|z| > a$

$$-a^n u[-n-1] \xrightarrow{Z} \frac{1}{1-az^{-1}}, \quad |z| < a$$

$$= Z\left\{ \left(\frac{1}{3}\right)^n u[n] \right\} + Z\left\{ 2^n u[-n-1] \right\} =$$

$$= \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-2z^{-1}} \quad , \quad |z| > 1/3 \;,\; |z| < 2$$

$$= \frac{1-2z^{-1} - 1 + \frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} \quad , \quad \frac{1}{3} < |z| < 2$$

$$\boxed{= \frac{\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} \quad , \quad \frac{1}{3} < |z| < 2 \quad = X(z)} \quad , (2)$$

. Correspondingly for the output it will be:

$$Y(z) = Z\{5(\tfrac{1}{3})^n u[n] - 5(\tfrac{2}{3})^n u[n]\} =$$

known pair ...

$$= 5 \cdot \frac{1}{1-\frac{1}{3}z^{-1}} - 5\frac{1}{1-\frac{2}{3}z^{-1}} \quad , \quad |z| > 1/3 \;,\; |z| > 2/3$$

$$\boxed{= \frac{\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})} \quad , \quad |z| > 2/3 \quad = Y(z)} \quad , (3)$$

$$(1)\ \overset{(2)}{\underset{(3)}{\Longrightarrow}} H(z) = + \frac{\frac{5}{3} z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)} \cdot \frac{\left(1-\frac{1}{3}z^{-1}\right)\left(1-2z^{-1}\right)}{\frac{5}{3}z^{-1}}, \quad |z| > 2/3$$

since the system is causal.

$$= \boxed{\frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}}, \quad |z| > 2/3 = H(z)}$$

ii) It can be simply given by:

$$h[n] = Z^{-1}\{H(z)\} = Z^{-1}\left\{\frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}}\right\} =$$

↳ We can usually break $H(z)$ into fractions using 2 ways: Partial Fraction Decomposition (PFD), or via Long Division*.

Here, PFD is unnecessary since:

$$= Z^{-1}\left\{\frac{1}{1-\frac{2}{3}z^{-1}} - 2\frac{z^{-1}}{1-\frac{2}{3}z^{-1}}\right\}, \quad |z| > 2/3$$

known pairs and properties

$$\boxed{= \left(\frac{2}{3}\right)^{n} u[n] - 2\left(\frac{2}{3}\right)^{n-1} u[n-1] = h[n]}$$

↳ there is a "simpler" form for $h[n]$, and it can be straightly given by the Long Division:

* if degree (denumerator) $\geq$ degree (numerator)

↳ With Long Division it would be:

$$\begin{array}{r|l} -2z^{-1}+1 & -\frac{2}{3}z^{-1}+1 \\ \hline -(-2z^{-1}+3) & 3 \\ \hline -2 & \end{array}$$

↳ So: $H(z) = 3 - \dfrac{2}{1-\frac{2}{3}z^{-1}}$

known pairs ...

$$\downarrow Z$$

$$\boxed{h[n] = 3\delta[n] - 2\left(\frac{2}{3}\right)^n u[n]}$$

iii) Like in the Fourier exercises:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}} \Leftrightarrow X(z)(1-2z^{-1}) = Y(z)\left(1-\frac{2}{3}z^{-1}\right)$$

$$\Leftrightarrow Y(z) - Y(z)\frac{2}{3}z^{-1} = X(z) - X(z)2z^{-1}$$

$$\swarrow z^{-1}$$

Z properties

$$\boxed{y[n] - \frac{2}{3}y[n-1] = x[n] - 2x[n-1]}$$

$$X(z)\cdot z^{-n_0} \xrightarrow{Z^{-1}} x[n-n_0]$$

END