Exercise 1

3º Φροντιστήριο ΗΥ-370
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- Input in LTI system: $x[n] = 2\cos\left(\frac{\pi n}{4}\right) + 8\sin\left(\frac{3\pi n}{4} - \frac{\pi}{5}\right)$

impulse responce : $h[n] = 4 \cdot \dfrac{\sin\left(\frac{(n-1)\pi}{2}\right)}{(n-1)\pi}$, output $y[n] = ?$

↳ known pair: $\dfrac{\sin(\omega_c n)}{\pi n} \overset{F}{\longleftrightarrow} H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$

↳ delay by 1 sample + scale by 4 : $H(e^{j\omega}) = \begin{cases} 4e^{-j\omega}, & |\omega| \leq \pi/2 \\ 0, & \text{elsewhere} \end{cases} = 4H_{lp}\, e^{-j\omega}$

↳ We know the form of the output given sinusoids in an LTI system

↳ ! Observation: $H(e^{j\omega})$ cuts off any frequencies not in $(-\pi/2, \pi/2)$, hence $8\sin\left(\frac{3\pi n}{4} - \frac{\pi}{5}\right)$ will be filtered out completely.

- $y[n] = 2\left|H(e^{j\pi/4})\right| \cos\left(\frac{\pi n}{4} + \measuredangle H(e^{j\pi/4})\right) =$

$$\boxed{\begin{aligned} &\cdot\ |H(e^{j\omega})| = 4 \\ &\cdot\ \measuredangle H(e^{j\omega}) = -\omega \end{aligned} \Bigg\}\ \text{for } |\omega| < \pi/2}$$

$= \underline{8\cos\left(\frac{\pi n}{4} - \frac{\pi}{4}\right) = y[n]}$

↳ input $\sum_{k=1}^{N} A_k \cos(\omega_k n + \theta_k)$ in LTI system, means:

↳ output $\sum_{k=1}^{N} A_k \left|H(e^{j\omega_k})\right| \cos\left(\omega_k n + \theta_k + \measuredangle H(e^{j\omega})\right)$

Exercise 2

(a') $\sum_{n=-\infty}^{+\infty} x[n] = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \Big|_{\omega=0} = X(e^{j\omega}) \Big|_{\omega=0} =$

$= \frac{3}{(1-0.8)^5} = \frac{3}{0.2^5} = 3\left(\frac{1}{0.2}\right)^5 = 3 \cdot 5^5$

(β') $\sum_{n=-\infty}^{+\infty} (-1)^n x[n] = \sum_{n=-\infty}^{+\infty} x[n] (e^{j\omega n}) \Big|_{\omega=\pi} = X(e^{j\omega}) \Big|_{\omega=\pi} =$

$= \cos^3(3\pi) = (-1)^3 = -1$

(γ') $\int_{-\pi}^{\pi} \frac{1}{1-0.3e^{-j\omega}} e^{j\omega} d\omega = \int_{-\pi}^{\pi} \frac{1}{1-0.3e^{-j\omega}} e^{j\omega n} d\omega \Big|_{n=1} =$

$= 2\pi (0.3)^n u[n] \Big|_{n=1} = 2\pi \cdot 0.3 \cdot 1 = 0.6\pi$

↳ for (a') we know $X(e^{j\omega}) = 3/(1-0.8e^{-j\omega})^5$

for (β') — " — $X(e^{j\omega}) = \cos^3(3\omega)$

Exercise 3

Causal LTI: $2y[n] - y[n-2] = x[n-1] + 3x[n-2] + 2x[n-3]$

(a') $H(e^{j\omega}) = ?$ $\quad \updownarrow F$

$$2Y(e^{j\omega}) - Y(e^{j\omega})e^{-j2\omega} = X(e^{j\omega})e^{-j\omega} + 3X(e^{j\omega})e^{-j2\omega} + 2X(e^{j\omega})e^{-j3\omega}$$

$$\Leftrightarrow \boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{e^{j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega}}{2 - e^{-j\omega}} = \frac{1}{2}e^{-j\omega}\frac{1 + 3e^{-j\omega} + 2e^{-j2\omega}}{1 - \frac{1}{2}e^{-j2\omega}}}$$

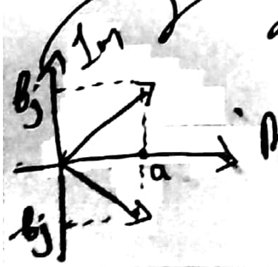
(b') $H_1(e^{j\omega}) = (e^{-j\omega} - 1/2)/(1 - 1/2\, e^{-j\omega})$, in series with

$h_2[n] = \delta[n] - \dfrac{\sin(\pi n/4)}{\pi n}$, total $|H(e^{j\omega})| = ?$

↳ In series means: $|H(e^{j\omega})| = |H_1(e^{j\omega})|\,|H_2(e^{j\omega})|$, (i)

$$\cdot\ |H_1(e^{j\omega})| = \left|\frac{e^{-j\omega} - \frac{1}{2}}{1 - \frac{1}{2}e^{-j\omega}}\right| = \frac{|e^{-j\omega} - \frac{1}{2}|}{|1 - \frac{1}{2}e^{-j\omega}|} = \frac{|e^{-j\omega}|\,|1 - \frac{1}{2}e^{j\omega}|}{|1 - \frac{1}{2}e^{-j\omega}|} =$$

$$= \frac{|1 - \frac{1}{2}e^{j\omega}|}{|1 - \frac{1}{2}e^{-j\omega}|} = 1$$, conjugates have the same norm, (ii)



$|a + Bj| =$
$|a - Bj|$

$\hookrightarrow H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/4 \\ 0, & \text{elsewhere} \end{cases} \overset{F^{-1}}{\longleftrightarrow} \dfrac{\sin(\pi n/4)}{\pi n}$ $\quad \delta[n] \overset{F}{\longleftrightarrow} 1$

$\hookrightarrow H_2(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$

$\hookrightarrow \boxed{|H_2(e^{j\omega})| = \begin{cases} 0, & |\omega| < \pi/4 \\ 1, & \text{elsewhere} \end{cases}}$ (iii)

$(i) \overset{(ii)}{\underset{(iii)}{\Rightarrow}} |H(e^{j\omega})| = |H_2(e^{j\omega})|$ 5

Exercise 4

LTI system: $y[n] - \frac{1}{9} y[n-2] = x[n-1]$, $y[-1] = 1$, $y[-2] = 0$, input: $x[n] = u[n]$, answer the following questions:

(α') Find the zero input response, $y_{zi}[n]$

(β') –//– impulse response, $h[n]$

(γ') –//– zero state response, $y_{zs}[n]$.

(δ') –//– total output of the system for $x[n]$ input, $y_t[n]$.

(ε') Discuss the stability of the system.

(a) characteristic polynomial :

$$y[n] - \frac{1}{9} y[n-2] = 0$$

$$\downarrow \qquad \downarrow$$

$$\gamma^2 - \frac{1}{9} = 0 \Leftrightarrow \gamma = \pm 1/3$$

- $y_{zi}[n] = \left(c_1 \left(\frac{1}{3}\right)^n + c_2 \left(-\frac{1}{3}\right)^n \right) u[n]$

- $y[-1] = 1 \Rightarrow 1 = 3c_1 - 3c_2$
- $y[-2] = 0 \Rightarrow 0 = 9c_1 + 9c_2$

$$\Rightarrow \begin{array}{r} 3 = 9c_1 - 9c_2 \\ 0 = 9c_1 + 9c_2 \quad + \\ \hline 3 = 18c_1 \Leftrightarrow c_1 = 1/6 \end{array}$$

- $c_2 = -c_1 = -1/6$, so:

- $y_{zi}[n] = \left(\frac{1}{6} \left(\frac{1}{3}\right)^n - \frac{1}{6} \left(-\frac{1}{3}\right)^n \right) u[n]$

(β') $h[n] = ?$ We know its form for systems like these:

$$h_0[n] = \left(d_1 \left(\frac{1}{3}\right)^n + d_2 \left(-\frac{1}{3}\right)^n \right) u[n] \quad , (i)$$

↳ we get d_1, d_2 by setting $x[n] = \delta[n]$:

$$y[n] - \frac{1}{9} y[n-2] = x[n]$$

$$h_0[n] - \frac{1}{9} h_0[n-2] = \delta[n]$$

$n=0$: $h_0[0] - \frac{1}{9} h[-2] = 1$, (ii)

$n=1$: $h_0[1] - \frac{1}{9} h[-1] = 0$, (iii)

$(i),(ii) \Rightarrow d_1 + d_2 = 1$

$(i),(iii) \Rightarrow \frac{1}{3}d_1 - \frac{1}{3}d_2 = 0$

$\Big\}$ $\underline{d_1 = d_2 = 1/2}$, so:

$$h_0[n] = \left(\frac{1}{2}\left(\frac{1}{3}\right)^n + \frac{1}{2}\left(-\frac{1}{3}\right)^n\right)u[n]$$

↳ In this system, the input is delayed by 1 sample ($x[n-1]$). Hence, the final resulting impulse response will also be delayed by 1 sample, i.e.: $h[n] = h_0[n-1] \Rightarrow$

$$\boxed{h[n] = \left(\frac{1}{2}\left(\frac{1}{3}\right)^{n-1} + \frac{1}{2}\left(-\frac{1}{3}\right)^{n-1}\right)u[n-1]}$$

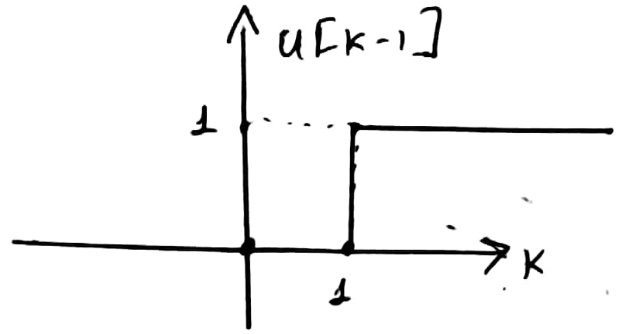
(γ') Zero state response -> initial conditions $= 0$, given by: (i)

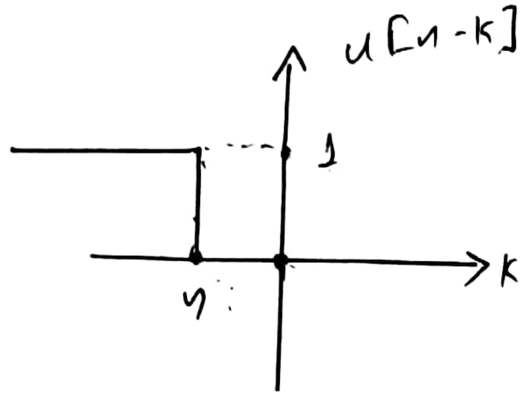
• $y_{zs}[n] = x[n] * h[n] = u[n] * \left(\left(\frac{1}{2}\left(\frac{1}{3}\right)^{n-1} + \frac{1}{2}\left(-\frac{1}{3}\right)^{n-1}\right)u[n-1]\right)$

• Convolution definition: $\sum_{k=-\infty}^{+\infty} h[k]x[n-k] = (h*x)[n]$

• For the first part it will be:

$$u[n] * \left(\frac{1}{2}\left(\frac{1}{3}\right)^{n-1}\right)u[n-1] = \sum_{k=-\infty}^{+\infty} \frac{1}{2}\left(\frac{1}{3}\right)^{k-1} u[k-1]\,u[n-k] =$$





↳ convolution will be non-zero only for $n \geq 1$:

$$= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{3}\right)^{k-1} = \frac{3}{2}\sum_{k=1}^{n}\left(1/3\right)^{k} \overset{*}{=} \frac{3}{2}\cdot\frac{\left(\frac{1}{3}\right)^{1}-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}}$$

$$= \frac{3}{2}\cdot\frac{3}{2}\left(\frac{1}{3}-\frac{1}{3}\left(\frac{1}{3}\right)^{n}\right) = \frac{3}{4}\left(1-\left(\frac{1}{3}\right)^{n}\right), \quad n \geq 1, \text{(ii)}$$

· For the second part it will be:

$$u[n] * \left(\frac{1}{2}\left(-\frac{1}{3}\right)^{n-1}\right)u[n-1] = \sum_{k=-\infty}^{+\infty}\frac{1}{2}\left(-\frac{1}{3}\right)^{k-1}u[k-1]u[n-k] =$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(-\frac{1}{3}\right)^{k-1} = -\frac{3}{2}\sum_{k=1}^{n}\left(-\frac{1}{3}\right)^{k} \overset{*}{=} -\frac{3}{2}\cdot\frac{\left(-\frac{1}{3}\right)^{1}-\left(-\frac{1}{3}\right)^{n+1}}{1-\left(-\frac{1}{3}\right)} =$$

$$= -\frac{3}{2}\cdot\frac{3}{4}\left(-\frac{1}{3}-\left(\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{n}\right)\right) =$$

$$= \frac{3}{8}\left(1-\left(-\frac{1}{3}\right)^{n}\right), \quad n \geq 1, \text{(iii)}$$

$*$

$$\sum_{n=N_1}^{N_2} a^{n} = \frac{a^{N_1}-a^{N_2+1}}{1-a}$$

Hence, in total, (i), (ii), (iii) $\Rightarrow$

$$y_{zs}[n] = (x * h)[n] = \left(\frac{3}{4}\left(1-\left(\frac{1}{3}\right)^n\right) + \frac{3}{8}\left(1-\left(-\frac{1}{3}\right)^n\right)\right)u[n-1]$$

(δ') Total system output is given by:

$y_t[n] = y_{zi}[n] + y_{zs}[n]$, known from previous questions

$$= \left(\frac{1}{6}\left(\frac{1}{3}\right)^n - \frac{1}{6}\left(-\frac{1}{3}\right)^n\right)u[n]$$

$$+\left(\frac{3}{4}\left(1-\left(\frac{1}{3}\right)^n\right) + \frac{3}{8}\left(1-\left(-\frac{1}{3}\right)^n\right)\right)u[n-1].$$

(ε') Since $|\gamma_i| < 1$, $\forall i \in \{1,2\}$, then we can conclude that this system is stable, i.e., all the absolute values of each root of the characteristic polynomial (question (a')) are less than one; a necessary and sufficient condition for stability.

Exercise 5

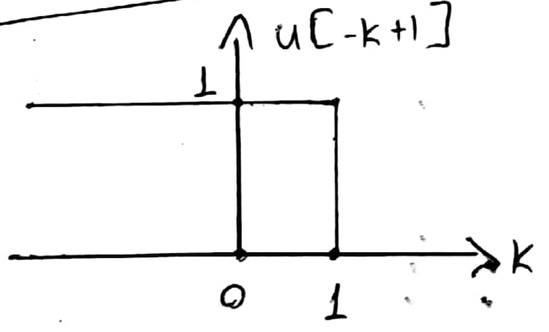
Convolve: $x[n] = \left(\frac{1}{2}\right)^n u[n-3]$, $y[n] = (-1)^{-n} u[-n+1]$.

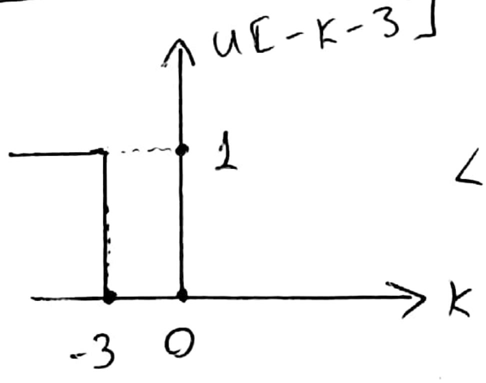
1^{st} way) Time domain analysis:

$$(y * x)[n] = \sum_{k=-\infty}^{+\infty} y[k]\, x[n-k] = \sum_{k=-\infty}^{+\infty} (-1)^{-k} u[-k+1] \left(\frac{1}{2}\right)^{n-k} u[n-k-3] =$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{+\infty} (-1)^{-k} \left(\frac{1}{2}\right)^{-k} u[n-k-3]\, u[-k+1]$$

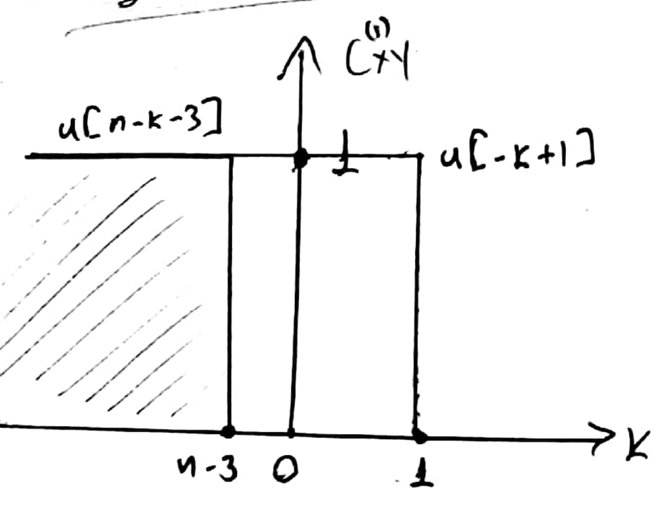
$$= \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{+\infty} \left(-\frac{1}{2}\right)^{-k} u[n-k-3]\, u[-k+1] \quad , \quad (i)$$



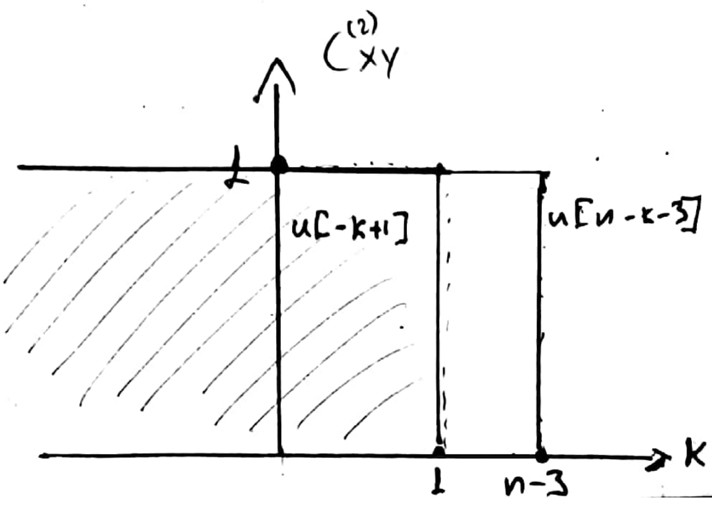


shifting this function by n will yield two cases:

1^{st} case: $n-3 < 1$



2^{nd} case: $n-3 \geq 1$



• For $n-3<1$, (i) $\Rightarrow$

$$(y*x)[n] = \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{n-3} \left(-\frac{1}{2}\right)^{-k} = \left(\frac{1}{2}\right)^n \sum_{k=3-n}^{+\infty} \left(-\frac{1}{2}\right)^k =$$

$$\overset{*}{=} \left(\frac{1}{2}\right)^n \frac{\left(-\frac{1}{2}\right)^{3-n}}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}\left(\frac{1}{2}\right)^n\left(-\frac{1}{2}\right)^{3-n} =$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n\left(-\frac{1}{2}\right)^3(-2)^n = -\frac{2}{3}\cdot\frac{1}{8}\cdot(-1)^n = \boxed{-\frac{1}{12}(-1)^n},$$

or $-\frac{1}{12}(-1)^n u[-n+3]$, (ii).

$$* \quad \sum_{n=N_1}^{+\infty} a^n = \frac{a^{N_1}}{1-a}$$

• For $n-3 \geq 1$, (i) $\Rightarrow$

$$(y*x)[n] = \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{1} \left(-\frac{1}{2}\right)^{-k} = \left(\frac{1}{2}\right)^n \sum_{k=-1}^{+\infty} \left(-\frac{1}{2}\right)^k =$$

$$\overset{*}{=} \left(\frac{1}{2}\right)^n \frac{\left(-\frac{1}{2}\right)^{-1}}{1-(-1/2)} = (-2)\frac{2}{3}\left(\frac{1}{2}\right)^n = \boxed{-\frac{4}{3}\left(\frac{1}{2}\right)^n},$$

or $-\frac{4}{3}\left(\frac{1}{2}\right)^n u[n-4]$, (iii).

In total: $(y*x)[n] = $ (i) + (ii) $\Rightarrow$

$$\Rightarrow \boxed{(x*y)[n] = -\frac{1}{12}(-1)^n u[-n+3] - \frac{4}{3}\left(\frac{1}{2}\right)^n u[n-4]}$$