Exercise 1

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Αιτιατό:
Causal

$$\boxed{x[n] = \delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3]}$$

(a') $X(e^{jw}) =$; Fourier Transform

- Known pair: $\delta[n] \overset{F}{\longleftrightarrow} 1$
- Time displacement property: $x[n-n_0] \overset{F}{\longleftrightarrow} e^{-jwn_0} \cdot X(e^{jw})$
- Linearity: $a x[n] + b y[n] \overset{F}{\longleftrightarrow} a X(e^{jw}) + b Y(e^{jw})$

$$F\{x[n]\} = X(e^{jw}) = 1 + 2 \cdot e^{-jw} - 2e^{-j2w} - e^{-j3w}$$

↳ simplify: show the sinusoids present/hidden, having in mind:
$\cos(x) = (e^{jx} + e^{-jx})/2$, $\sin(x) = (e^{jx} - e^{-jx})/2j$

$$\cdot X(e^{jw}) = 1 - e^{-j3w} + 2e^{-jw} - 2e^{-j2w}$$

$$= 2j\left(\frac{1-e^{-j3w}}{2j} + 2\left(\frac{e^{-jw} - e^{-j2w}}{2j}\right)\right)$$

$$= 2j\left(e^{-j\frac{3w}{2}}\left(\frac{e^{j\frac{3w}{2}} - e^{-j\frac{3w}{2}}}{2j}\right) + 2e^{-j\frac{3w}{2}}\left(\frac{e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}}{2j}\right)\right)$$

$$= 2je^{-j\frac{3w}{2}}\left(\sin(3w/2) + 2\sin(w/2)\right)$$

↳ helpful representation for the follow-up questions

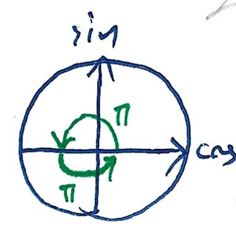
(B') Find & (roughly) Draw (qualitatively) the spectrum $|X(e^{j\omega})|$

- $|X(e^{j\omega})| = \left| 2j e^{-j3\omega/2} \left(\sin(3\omega/2) + 2\sin(\omega/2) \right) \right|$

$$= |2| \cdot |j| \cdot |e^{-j3\omega/2}| \cdot |\sin(3\omega/2) + 2\sin(\omega/2)|$$

It is always true that $|e^{jx}| = 1$, because:

$$|e^{jx}| = |\cos(x) + j\sin(x)| = \sqrt{\cos^2(x) + \sin^2(x)} = \sqrt{1} = 1$$



$$= 2 \cdot 1 \cdot 1 \cdot |\sin(3\omega/2) + 2\sin(\omega/2)| =$$

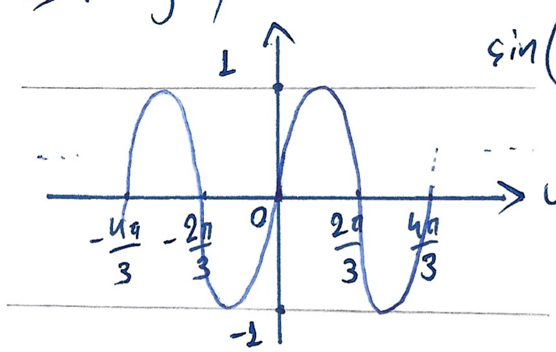
$$= 2|\sin(\omega/2) + 2\sin(\omega/2)| = |X(e^{j\omega})|$$

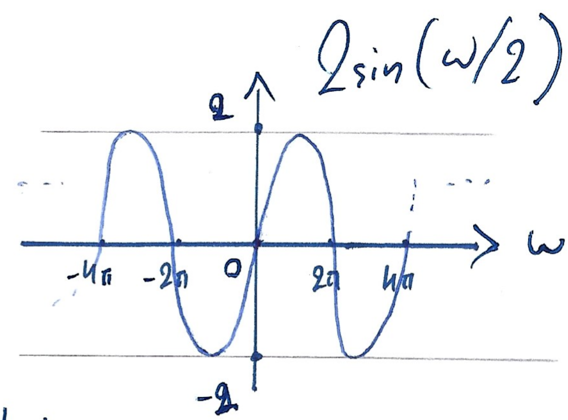
↳ Tips on how to roughly draw functions like these:

- $\sin(3\omega/2) = 0 \Leftrightarrow \frac{3\omega}{2} = k\pi \Leftrightarrow \omega = 2k\pi/3, \; k \in \mathbb{Z}$
- $\sin(\omega/2) = 0 \Leftrightarrow \frac{\omega}{2} = k\pi \Leftrightarrow \omega = 2k\pi, \; k \in \mathbb{Z}$

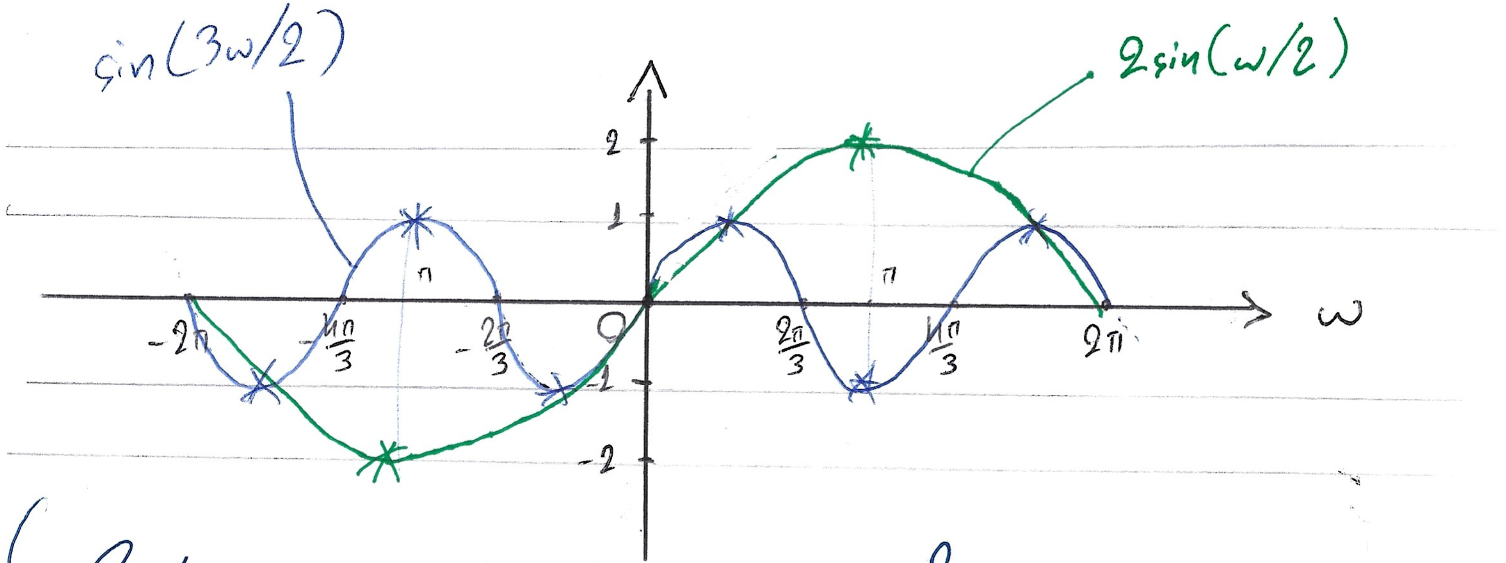
zero points

↳ Roughly draw the sinusoids individually:

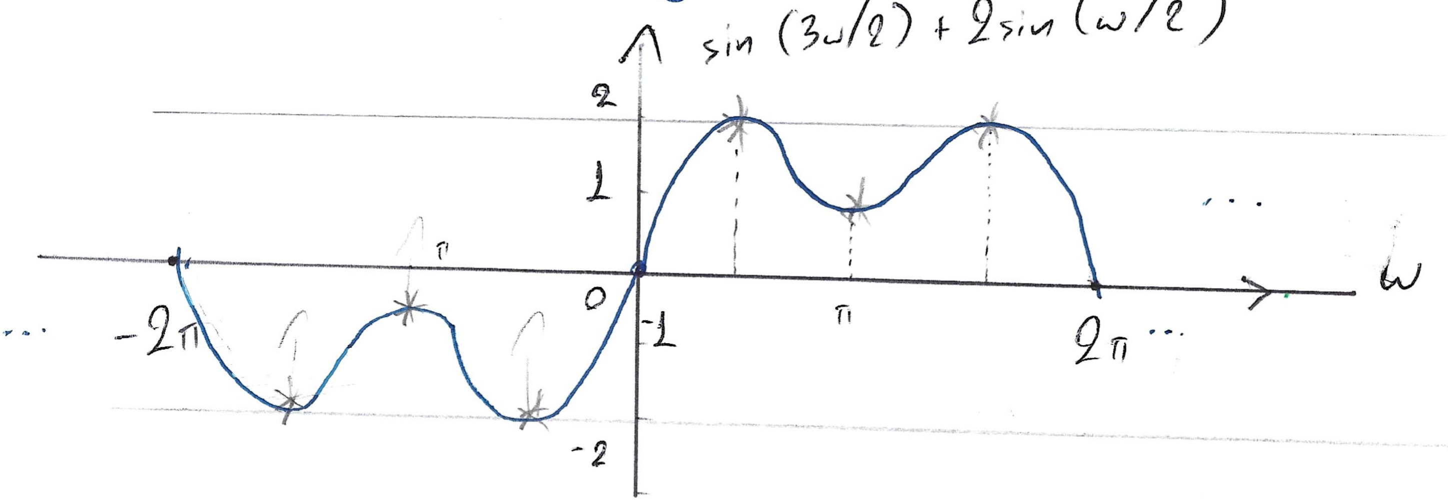


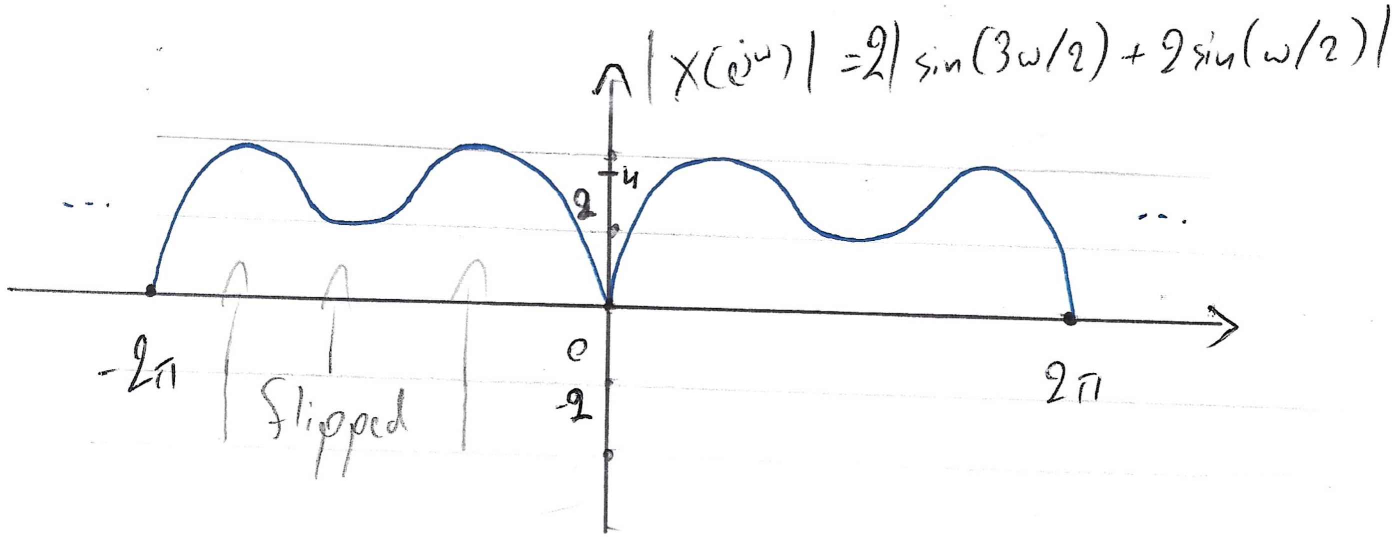


or in the same plot



↳ 3 bumps + sign changes every 2π:





(γ') Find and (roughly) draw the phase spectrum $\measuredangle X(e^{j\omega})$

↳ multiplication → addition of separate phases: $\measuredangle e^{j\varphi} = \varphi$, $\measuredangle a = \measuredangle a e^{j0} = 0$, $a \in \mathbb{R}$

$$\measuredangle X(e^{j\omega}) = \measuredangle 2 + \measuredangle j + \measuredangle e^{-j3\omega/2} + \measuredangle\left(\sin\left(\frac{3\omega}{2}\right) + 2\sin\left(\frac{\omega}{2}\right)\right)$$

$$= \measuredangle 2e^{j0} + \measuredangle e^{j\pi/2} + \measuredangle e^{-j3\omega/2} + \measuredangle\underbrace{\left(\sin\left(\frac{3\omega}{2}\right) + 2\sin\left(\frac{\omega}{2}\right)\right)}$$

↳ phase is expressed in [-π, π]

↳ we determine the phase of a real function by looking at its sign

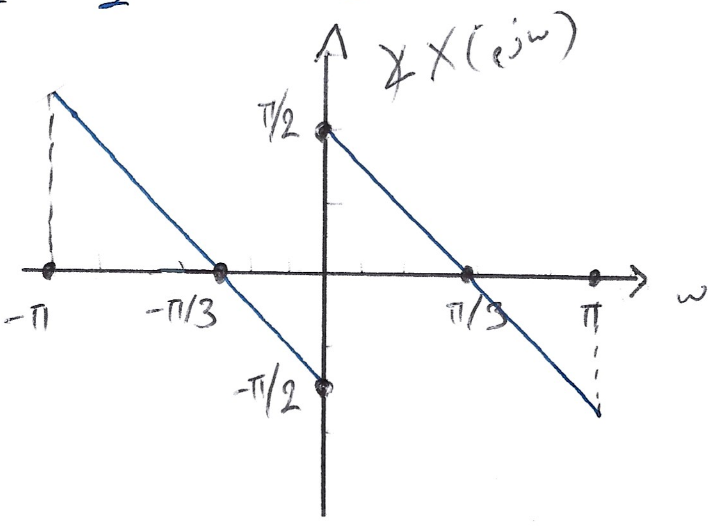
↳ positive ⇒ $\varphi = 0$, negative + ω positive ⇒ $\varphi = \pi$ (by convention)

negative + ω negative ⇒ $\varphi = -\pi$ (also by convention)

↳ In [0, π] the sine sum is positive, hence its $\varphi = 0$

↳ real signal ⇒ add symmetry in (-π, π] so we can also draw it in (-π, 0]

$$= 0 + \frac{\pi}{2} - \frac{3\omega}{2} + 0 = \frac{\pi}{2} - \frac{3\omega}{2} \text{ in } [0, \pi]$$



Exercise 2

• LTIs connected in parallel with $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$, $h_2[n]$, and total frequency response of the system:

$$\boxed{H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}}\ ;(i),$$ find $h_2[n]$.

↳ In parallel means: $H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega}) \Leftrightarrow$

$$\Leftrightarrow H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega}) \quad ,(ii)$$

↳ $\boxed{H_1(e^{j\omega})}$ is given by the known pair ...:

$$a^n u[n] \xrightarrow{F} \frac{1}{1 - ae^{-j\omega}}, |a|<1$$

$$= \boxed{\frac{1}{1 - \frac{1}{3}e^{-j\omega}}} \quad ,(iii)$$

$$(ii) \underset{(iii)}{\overset{(i)}{\Longrightarrow}} H_2(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}} - \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

↳ likewise, we want to simplify, but now by factorizing the denominator of the first fraction

↳ trick: $e^{-j\omega} = x$, so: $x^2 - 7x + 12 = 0$

$\Delta = b^2 - 4ac = 49 - 48 = 1$ $\qquad (x-3)(x-4) = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{7 \pm 1}{2}, \quad \boxed{x_1 = 4, \ x_2 = 3}$$

$$= \frac{-12+5e^{-j\omega}}{(e^{-j\omega}-3)(e^{-j\omega}-4)} - \frac{1}{1-\frac{1}{3}e^{-j\omega}} =$$

↳ make the denominator of the second fraction appear in the first fraction

$$= \frac{-12+5e^{-j\omega}}{3\left(\frac{1}{3}e^{-j\omega}-1\right)4\left(\frac{1}{4}e^{-j\omega}-1\right)} - \frac{1}{1-\frac{1}{3}e^{-j\omega}} =$$

$$= \frac{-12+5e^{-j\omega}}{12\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)} - \frac{12\left(1-\frac{1}{4}e^{-j\omega}\right)}{12\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)} =$$

$$= \frac{-1+\frac{5}{12}e^{-j\omega}-1+\frac{1}{4}e^{-j\omega}}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)} =$$

$$= \frac{-2+\frac{2}{3}e^{-j\omega}}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)} = \frac{-2\left(1-\frac{1}{3}e^{-j\omega}\right)}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \boxed{-2\frac{1}{1-\frac{1}{4}e^{-j\omega}} = H_2(e^{j\omega})} \xrightarrow[\text{known pair}]{F^{-1}} h_2[n] = -2\left(\frac{1}{4}\right)^n u[n]$$

Exercise 3

Stable & causal LTI system with input-output pair:

$$\left(\frac{4}{5}\right)^n u[n] \longrightarrow n\left(\frac{4}{5}\right)^n u[n]$$

(a') Frequency response of the system $H(e^{j\omega}) = ?$

↳ we know: $\boxed{H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega})}$, (i)

$\cdot\, x[n] = \left(\frac{4}{5}\right)^n u[n]$, $y = n \cdot x[n]$

known pair $\updownarrow F$ frequency differenciation property $\updownarrow F$

$$\boxed{X(e^{j\omega}) = \frac{1}{1-\frac{4}{5}e^{-j\omega}}} \text{, (ii)}, \quad Y(e^{j\omega}) = j\frac{d}{d\omega}\left(X(e^{j\omega})\right) \Bigg\} \overset{(ii)}{\Longrightarrow}$$

$$Y(e^{j\omega}) = j\frac{d}{d\omega}\left(\frac{1}{1-\frac{4}{5}e^{-j\omega}}\right) =$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{1}{f^2(x)} \cdot f'(x)$$

$$= j\left(\frac{-1}{1-\frac{4}{5}e^{-j\omega}}\right)^2 \cdot \frac{d}{d\omega}\left(1-\frac{4}{5}e^{-j\omega}\right) =$$

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$$

$$= j\,\frac{1}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2} \cdot \left(\frac{4}{5}e^{-j\omega}\right) \cdot (-j) =$$

$$= \boxed{\frac{4}{5}e^{-j\omega} \Big/ \left(1-\frac{4}{5}e^{-j\omega}\right)^2 = Y(e^{j\omega})} \text{ , (iii)}$$

$$(i) \overset{(ii)}{\underset{(iii)}{\Rightarrow}} H(e^{j\omega}) = \frac{\frac{4}{5}e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2} \cdot \frac{1-\frac{4}{5}e^{-j\omega}}{1} \Leftrightarrow$$

$$\Leftrightarrow \boxed{H(e^{j\omega}) = \frac{4}{5}\frac{e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}}}$$

(B') Difference equation that describes this system

$$\cdot H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{4}{5}\cdot\frac{e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}} \Leftrightarrow$$

$$\Leftrightarrow Y(e^{j\omega})\left(1-\frac{4}{5}e^{-j\omega}\right) = X(e^{j\omega})\left(\frac{4}{5}e^{-j\omega}\right) \Leftrightarrow$$

$$\Leftrightarrow Y(e^{j\omega}) - Y(e^{j\omega})\frac{4}{5}e^{-j\omega} = X(e^{j\omega})\frac{4}{5}e^{-j\omega}$$

$\updownarrow F^{-1}$ difference in time property $\updownarrow F^{-1}$ $\updownarrow F^{-1}$

$$y[n] - \frac{4}{5}y[n-1] = x[n-1]$$

Exercise 4

LTI system described by: $y[n] = x[n] + x[n-10]$

(a') Calculate & draw amplitude & phase responses using:

$$e^{ja} + e^{-jb} = e^{j(a-b)/2} \cdot \left(e^{j(a+b)/2} + e^{-j(a+b)/2}\right) \quad ,(i)$$

↳ frequency response: $H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega})$, (ii)

$$y[n] = x[n] + x[n-10]$$

$$\Updownarrow F \qquad \Updownarrow F \qquad \Updownarrow F$$

$$Y(e^{j\omega}) = X(e^{j\omega}) + X(e^{j\omega}) \cdot e^{-j10\omega} = X(e^{j\omega})(1 + e^{-j10\omega}) \overset{(ii)}{\Longrightarrow}$$

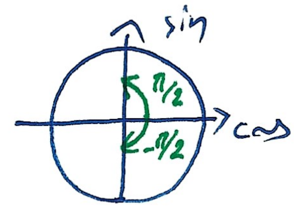
$$\boxed{H(e^{j\omega}) = 1 + e^{-j10\omega}}$$, and using the property:

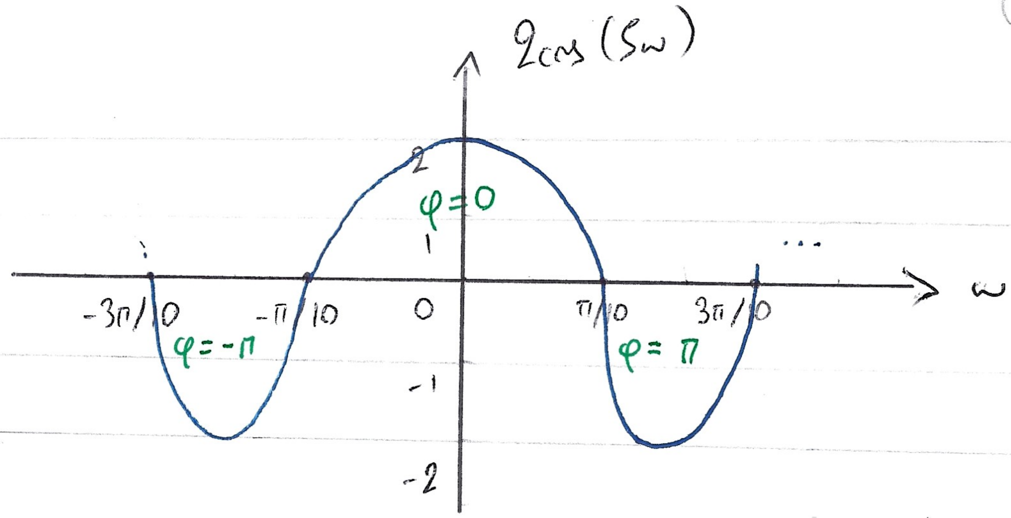
$$= e^{-j0\omega} + e^{-j10\omega} \overset{(i)}{=} e^{j(-10\omega)/2} \cdot \left(e^{j(10\omega)/2} + e^{-j(10\omega)/2}\right)$$

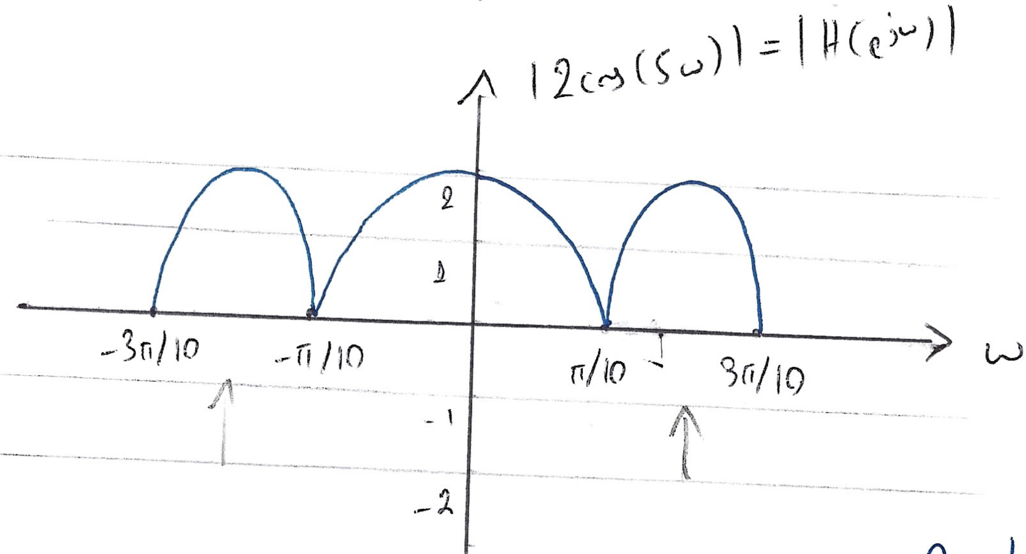
$$= e^{-j5\omega}\left(e^{j5\omega} + e^{-j5\omega}\right) = \boxed{2e^{-j5\omega}\cos(5\omega) = H(e^{j\omega})}$$

- $|H(e^{j\omega})| = |2| \cdot |e^{-j5\omega}| \cdot |\cos(5\omega)| = \boxed{2|\cos(5\omega)| = |H(e^{j\omega})|}$
- $\measuredangle H(e^{j\omega}) = \measuredangle 2 + \measuredangle e^{-j5\omega} + \measuredangle \cos(5\omega) = \boxed{-5\omega + \measuredangle\cos(5\omega) = \measuredangle H(e^{j\omega})}$

• $\cos(5\omega) = 0 \Leftrightarrow 5\omega = 2k\pi \pm \frac{\pi}{2} \Leftrightarrow \omega = \frac{2k\pi}{5} \pm \frac{\pi}{10}$





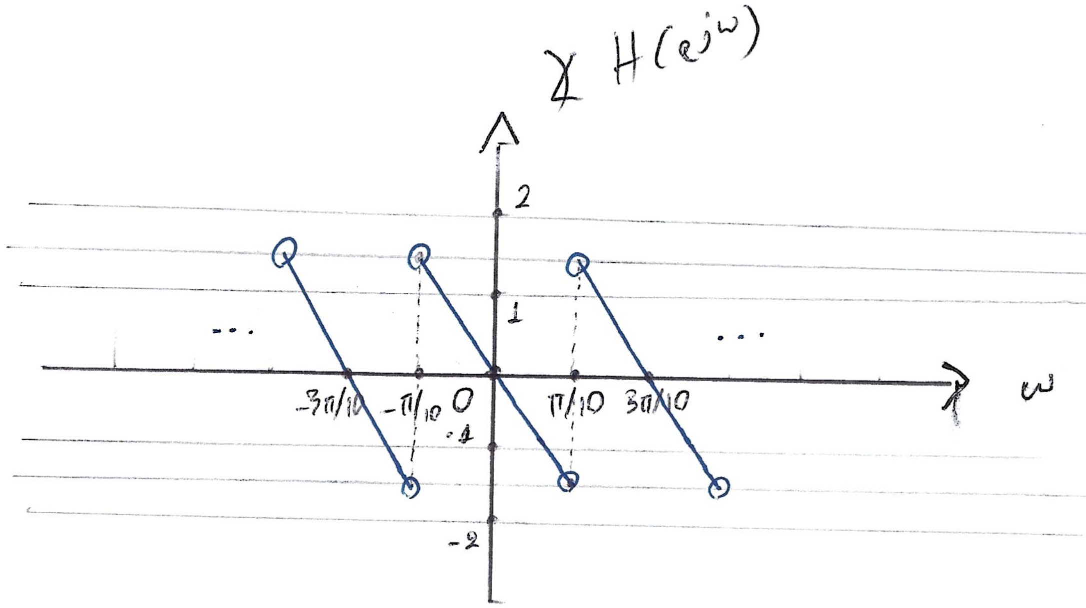


↳ for the phase, again $\measuredangle \cos(5\omega)$

- → 0, when $\cos(5\omega) > 0$
- → π, -"- $\cos(5\omega) < 0$, $\omega > 0$
- → $-\pi$, -"- "- , $\omega < 0$

↳ its sign changes every $\frac{2k\pi}{5} \pm 10$, so:

$$\measuredangle \cos(5\omega) = \begin{cases} \vdots \\ -\pi, & -3\pi/10 < \omega < -\pi/10 \\ 0, & -\pi/10 < \omega < \pi/10 \\ \pi, & \pi/10 < \omega < -\pi/10 \\ \vdots \end{cases}$$



(β') Find the output for input: i. $x[n] = \cos\left(\frac{\pi n}{10}\right) + 3\sin\left(\frac{\pi n}{3} + \frac{\pi}{10}\right)$

and for input: ii. $x[n] = 10 + 5\cos\left(\frac{2\pi n}{5} + \frac{\pi}{10}\right)$

↳ We know an LTI's system output form given sinusoidal sums in the input, and it is:

$$x[n] = \sum_{k=1}^{N} A_k \cos(\omega_k n + \theta_k)$$, input, then output:

↳ $$y[n] = \sum_{k=1}^{N} A_k \left|H(e^{j\omega_k})\right| \cos\left(\omega_k n + \measuredangle H(e^{j\omega_k})\right)$$, so:

i. $y[n] = \left|H(e^{j\pi/10})\right| \cos\left(\frac{\pi n}{10} + \measuredangle H(e^{j\pi/10})\right) +$

$3\left|H(e^{j\pi/3})\right| \sin\left(\frac{\pi n}{3} + \frac{\pi}{10} + \measuredangle H(e^{j\pi/3})\right)$

$= 2\left|\cos\left(5 \cdot \frac{\pi}{10}\right)\right| \cdot \cos\left(\frac{\pi n}{10} + \measuredangle H(e^{j\pi/10})\right) +$

$+ 3 \cdot 2 \cdot |\cos(5\pi/3)| \cdot \sin\left(\frac{\pi n}{3} + \frac{\pi}{10} - \frac{5\pi}{3}\right) = \boxed{3\sin\left(\frac{\pi n}{3} - \frac{47\pi}{30}\right)}$

ii. ↳ We also know the output for exponential inputs:

$$x[n] = \sum_{k=1}^{N} A_k e^{j(\omega_k n + \theta_k)}$$, input, then output:

$$y[n] = \sum_{k=1}^{N} A_k H(e^{j\omega_k}) e^{j(\omega_k n + \theta_k)}$$, so

↳ we treat 10 as if it is $10e^{j0}$, so

$$y[n] = 10\, H(e^{j0})\, e^{j0} + 5\, |H(e^{j2\pi/5})| \cos\left(\frac{2\pi n}{5} + \frac{\pi}{2} + \measuredangle H(e^{j2\pi/5})\right)$$

$$= 10 \cdot 2 \cdot 1 + 5 \cdot 2 \cdot \left|\cos\left(5 \cdot \frac{2\pi}{5}\right)\right| \cos\left(\frac{2\pi n}{5} + \frac{\pi}{2} - 5 \cdot \frac{2\pi}{5} + 0\right)$$

$$= \boxed{20 + 10\cos\left(\frac{2\pi n}{5} + \frac{\pi}{2}\right)}$$

ΤΕΛΟΣ.