Assistant Lecture 9 - Final 2017
CS370
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Exercise 1

Assume LTI System with transfer function.

$$H(z) = \frac{4 + 2z^{-1} - \frac{1}{2}\cdot z^{-2}}{1 - \frac{1}{4}\cdot z^{-2}}$$

i) Assuming the system is stable, find $h[n]$

ii) Implement a graph using 3 multipliers and 2 memory slots. Multiplying by ± 1 doesn't count.

Solution:

i) Long polynomial division.

$$H(z) = \frac{4 + 2z^{-1} - \frac{1}{2}\cdot z^{-2}}{1 - \frac{1}{4}\cdot z^{-2}} = 2 + \frac{2 + 2z^{-1}}{1 - \frac{1}{4}\cdot z^{-2}}$$

Applying simple fraction decomposition:

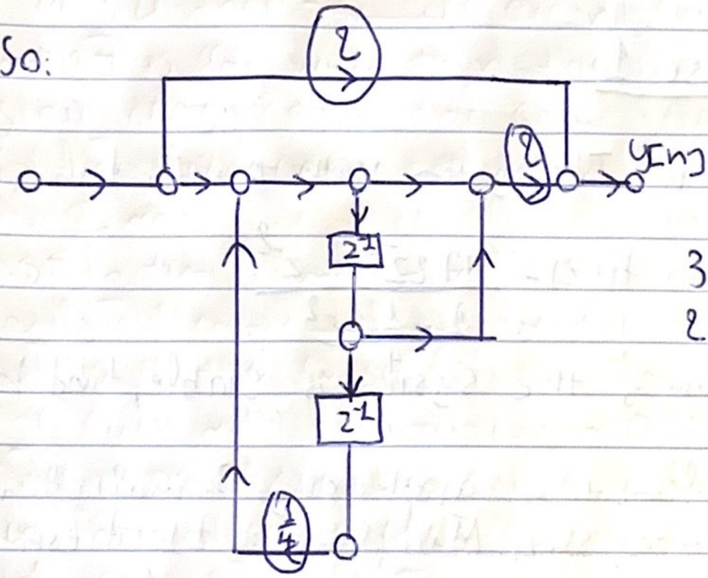
$A = 3$, $B = -1$, so:

$$H(z) = 2 + 3\cdot\frac{1}{1 - \frac{1}{2}\cdot z^{-1}} - \frac{1}{1 + \frac{1}{2} z^{-1}} \overset{Z^{-1}\{\}}{\Rightarrow} h[n] = 2\delta[n] + 3\cdot\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{2}\right)^n u[n]$$

Since the system is stable, ROC: $|z| > \frac{1}{2}$ (includes unit circle)

ii)
$$H(z) = 2 + \frac{2+2z^{-1}}{1-\frac{1}{4}z^{-1}} = 2 + 2\,\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}$$

So:



3 multipliers
2 memory slots.

Exercise 2

In figure 1, you will see an LTI System with 1 input and 1 output. You will also see the input $x[n]$, the amplitude spectrum $|X(e^{j\omega})|$, the amplitude reponse $|H(e^{j\omega})|$ and the group delay $\tau_g(e^{j\omega})$ of the system and its amplitude spectrum, $|Y(e^{j\omega})|$.

Assume input:
$$x[n] = w[n-50]\cdot(\cos(0.2n\cdot\pi) + \cos(0.8\pi\cdot n)) + w[n-230]\cdot\cos(0.4\cdot\pi\cdot n)$$
with $w[n]$ a window with length $= 41$ samples, so $w[n] \neq 0$, $0 \leq n \leq 40$. Explain the output behavior both in time and frequency domains.

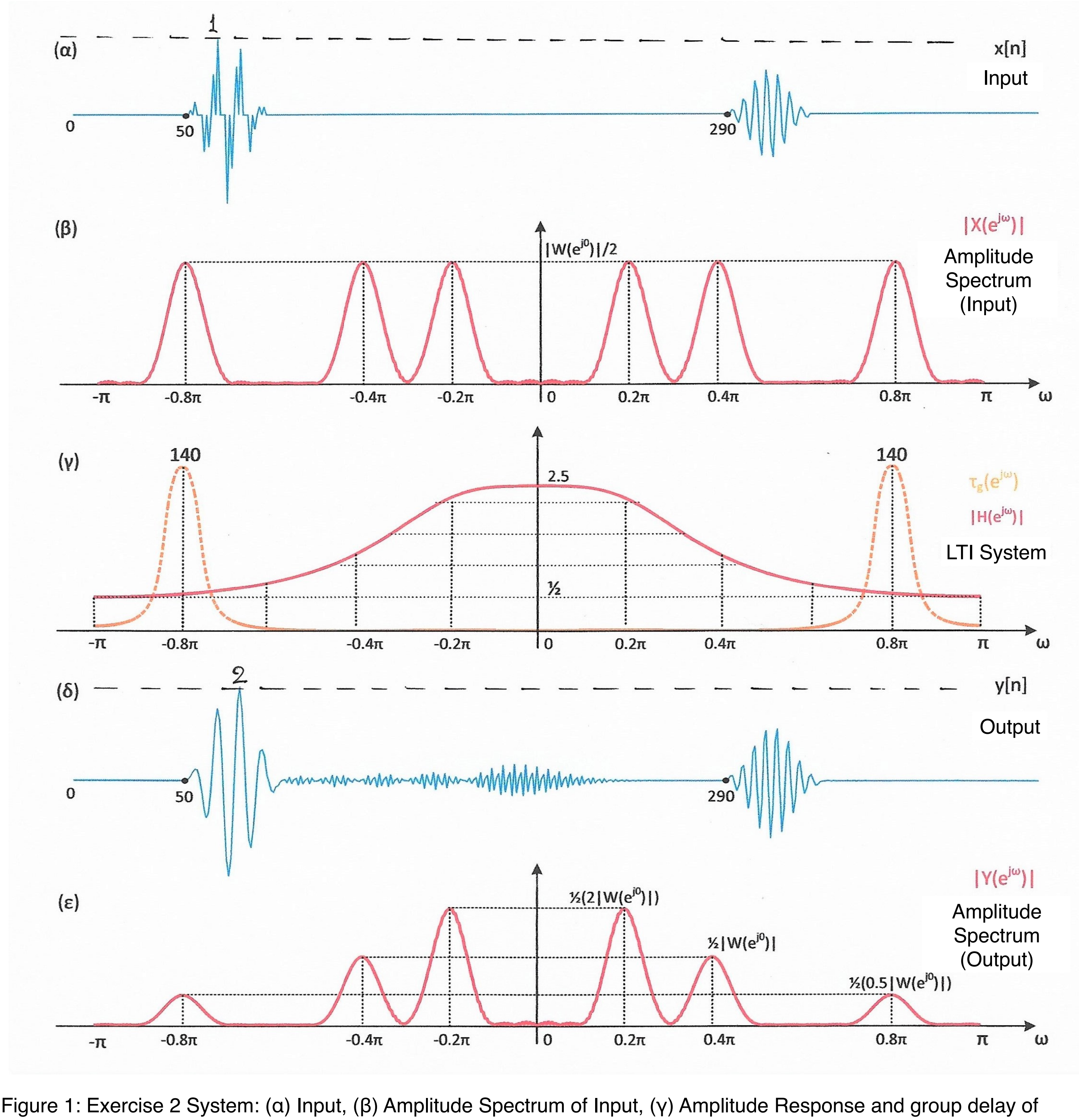


Figure 1: Exercise 2 System: (α) Input, (β) Amplitude Spectrum of Input, (γ) Amplitude Response and group delay of LTI system, (δ) output, (ε) Amplitude Spectrum of Output

Solution

we observe that in input, the frequencies $w=0.2\pi$, $w=0.8\pi$ are added under the same window - they exist in the same space, 50 samples from the starting point ($n=0$). The input signal will be affected in both amplitude and phase spectrums on the output:

- On its amplitude, the input coefficients are affected as following:

  $>0.2\pi$: The amplitudes of this frequency group will be multiplied by 2

  $>0.4\pi$: multiplied by approximately 1, so they will remain almost the same

  $>0.8\pi$: multiplied by $\frac{1}{2}$.

The subfigure (2) confirms our observations.

- On its phase, we know that for narrow band signals - essentially gathered around a frequency w_0 - the group delay informs us for the delay of the output w.r.t. the input. Observing the group delay we see that only the coefficients around the frequency 0.8π are subject to delays.

However, the pulse with frequency $w=0.8\pi$ is not narrow band as it has a wide central lobe (approx. in range $(0.7\pi, 0.9\pi)$) - we can confirm by comparing the group delay at $\omega = 0.8\pi$ with the amplitude spectrum of the input at the respective range.

we see that the lobe around frequency $\omega = 0.8\pi$ is far wider than the lobe of the group delay around the same frequency, So the frequencies within $(0.7\pi, 0.9\pi)$ - all of which are part of the pulse with frequency 0.8π in time domain - will subject to a significantly different delay from the rest of its group.

This is the cause of the variance of the pulse with frequency $\omega = 0.8\pi$ on the output, as observed in subfigure (8). We can see that different parts of the pulse are delayed differently. The other frequencies have 0 group delay, so they won't be moved in time.

Exercise 3

A stable LTI System has the following transfer function:

$$H(z) = \frac{\left(1 - \frac{3}{2} \cdot z^{-1}\right)\left(1 - \frac{1}{3} \cdot z^{-1}\right)}{1 + z^{-1} + \frac{1}{2} \cdot z^{-2}}$$

Find a stable and Causal System $G(e^{j\omega})$ So that:

$$|G(e^{j\omega}) H(e^{j\omega})| = 1 \Rightarrow$$
$$|H(e^{j\omega})| = \frac{1}{|G(e^{j\omega})|}$$

Solution

Essentially, we are looking for a stable causal system $G(e^{j\omega})$ so that $|G(e^{j\omega})| = |H^{-1}(e^{j\omega})|$

The given $H(z)$ has many inverse filters, but only 1 is causal and stable: $H^{-1}_{min}(z)$!

So:

$$H(z) = \frac{(1-\frac{3}{2}z^{-1})(1-\frac{1}{3}z^{-1})}{1+z^{-1}+\frac{1}{2}z^{-2}}$$

Poles: $z = -\frac{1}{2} \pm j\cdot\frac{1}{2}$ (within unit circle)

Zeros: $z = \frac{3}{2}$, $z = \frac{1}{3}$. We need to "mirror" the zero $z = \frac{3}{2}$ within the unit circle to get a min phase system. So, $H_{min}(z)$:

$$H_{min}(z) = \frac{(z^{-1}-\frac{3}{2})(1-\frac{1}{3}z^{-1})}{1+z^{-1}+\frac{1}{2}z^{-2}} = -\frac{3}{2}\,\frac{(1-\frac{2}{3}z^{-1})(1-\frac{1}{3}z^{-1})}{1+z^{-1}+\frac{1}{2}\cdot z^{-2}}$$

So, $$G(z)\Big|_{z=e^{j\omega}} = \frac{1}{H_{min}(e^{j\omega})} = -\frac{2}{3}\,\frac{1+e^{-j\omega}+\frac{1}{2}\cdot e^{-j2\omega}}{(1-\frac{2}{3}e^{-j\omega})(1-\frac{1}{3}\cdot e^{-j\omega})}$$

Exercise 4

For an LTI system, you are given the following info.

i) The system has (generalized) linear phase

ii) The system completely cuts off frequency $w_0 = \pm \frac{\pi}{3}$

iii) The amplitude response of the system equals 1 for $w=0$ and $w=\pi$

Find the system with the minimum possible duration in samples N. What type of linear phase is it?

Solution

Since the system has generalized linear phase, it is also an FIR, so all poles will be at 0, and all zeros will be on the ~~imaginary~~ complex plane. To completely cut off frequencies $w = \pm \frac{\pi}{3}$, we need zeros on the unit circle on those frequencies: $(1 - e^{-j\frac{\pi}{3}} z^{-1})(1 - e^{+j\frac{\pi}{3}} z^{-1})$

So, the frequency response will be

$$H(e^{jw}) = A \cdot (1 - e^{j\frac{\pi}{3}} \cdot z^{-1})(1 - e^{-j\frac{\pi}{3}} \cdot z^{-1})\Big|_{z = e^{jw}} = A \cdot (1 - e^{-jw} + e^{-j2w})$$

for $w=0$, $|H(e^{0})| = A$, but for $w=\pi$, $|H(e^{j\pi})| = 3A$

The problem $\begin{matrix} A=1 \\ 3A=1 \end{matrix}$ has no solutions!

We need to add 1 more term $1+Bz^{-1}+z^{-2}$ - a linear phase term that minimizes the system's order.

So: $H(z) = A\cdot(1-z^{-1}+z^{-2})(1+Bz^{-1}+z^{-2})$

where: $\left.\begin{matrix} |H(e^{j0})| = A\cdot(2+B) = 1 \\ |H(e^{j\pi})| = 3A(2-B) = 1 \end{matrix}\right\} A = \frac{1}{3},\ B=1$

So the final system:

$$H(z) = \frac{1}{3}\cdot(1+z^{-2}+z^{-4})$$

Since $M=4$ and the symmetry center is $n=2$, its a type I linear phase.

Exercise 5

The causal system:

$$H_1(z) = \frac{3}{1-\frac{1}{2}z^{-1}}$$

is connected in parallel with the anti-causal system:

$$H_2(z) = \frac{2}{1-2z^{-1}}$$

Calculate the $\int_{-\pi}^{\pi} H(e^{j\omega})\,d\omega$

where $H(e^{j\omega})$

the overall frequency response of the system

Solution

Parallel connection $\Rightarrow$

$$H(z) = H_1(z) + H_2(z) \Leftrightarrow h[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\cdot(2)^n\cdot u[-n-1]$$

$$\text{So: } \int_{-\pi}^{\pi} H(e^{j\omega})\,d\omega = \int_{-\pi}^{\pi} H(e^{j\omega})\cdot e^{j\omega 0}\,d\omega = 2\pi\cdot h[n]\Big|_{n=0}$$

$$= 2\pi\cdot 3 = 6\pi$$

Exercise 6

A System is described by the following equation.

$$Y(e^{j\omega}) = e^{-j\omega} X(e^{j\omega}) + \frac{\partial}{\partial\omega} X(e^{j\omega})$$

i) Compute the system's response for input $x[n] = \delta[n]$

ii) Check if the system is time invariant and stable.

Solution

i) $x[n] = \delta[n] \Leftrightarrow X(e^{j\omega}) = 1$

Therefore

$$Y(e^{j\omega}) = e^{-j\omega} \cdot 1 + \frac{\partial}{\partial\omega} 1 = e^{-j\omega}$$

So: $y[n] = \delta[n-1]$

ii) In time domain, the given equation is expressed as:

$$y[n] = x[n-1] - jn\,x[n]$$

due to n variable as a coefficient of x, the system is not time invariant. (Prove it!)
For the same reason, it is unstable since $n \to \infty$, $|y[n]| \to \infty$ (prove it!)

Exercise 7

Let a causal LTI system:

$$H(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

for which you know it is unstable. Modifying the system so that $h'[n] = \lambda^n h[n]$,

Show that picking the right λ we get a new stable system.

Solution

Refactoring: $H(z) = \frac{1}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$, $z = d_k$ the system's poles

It is unstable, so at least 1 pole is outside the unit circle. Using Fourier transform properties, the new system has transport function

$$H'(z) = H(\lambda^{-1} z) = \frac{1}{\prod_{k=1}^{N}(1 - d_k \lambda \cdot z^{-1})}$$

Let $d_i = |d_i| \cdot e^{j\theta_i}$ the poles with $|d_i| > 1$ that make the system unstable. In $h'[n]$, they are expressed as $\lambda \cdot d_i$, where $|\lambda d_i| > 1$.
If $\lambda < \frac{1}{|d_{max}|}$, then the causal system $H'(z)$ will be stable