8º Φροντιστήριο

Exercise 1 (From 2016 final exams)

An anti-causal system has the following transfer function:

$$H(z) = \frac{2 - 5z^{-1} + 2z^{-2}}{2 - z^{-1} - z^{-2}}$$

(α') What is the ROC (Region of Convergence)?

Ans

The poles are located at: $2 - z^{-1} - z^{-2} = 0 \Leftrightarrow z_1 = 1,\ z_2 = -\frac{1}{2}$,

and since the system is anti-causal, the ROC will be $|z| < 1/2$.

(β') Find the impulse response $h[n]$ of this system.

Ans

Numerator degree $(H(z)) \geq$ denumerator degree: (long division)

$$\begin{array}{r|l} 2z^{-2} - 5z^{-1} + 2 & -z^{-2} - z^{-1} + 2 \\ \hline -2z^{-2} - 2z^{-1} + 4 & -2 \\ \hline -7z^{-1} + 6 & \end{array}$$

$$H(z) = -2 + \frac{-7z^{-1} + 6}{-z^{-2} - z^{-1} + 2}$$

PFD: $\dfrac{-7z^{-1}+6}{-z^{-2}-z^{-1}+2} = \dfrac{-7z^{-1}+6}{2(1-z^{-1})(1+\frac{1}{2}z^{-1})} = \dfrac{-\frac{7}{2}z^{-1}+3}{(1-z^{-1})(1+\frac{1}{2}z^{-1})} = \dfrac{A}{1-z^{-1}} + \dfrac{B}{1+\frac{1}{2}z^{-1}}$

$\Leftrightarrow 3 - \frac{7}{2} z^{-1} = A\left(1 + \frac{1}{2} z^{-1}\right) + B\left(1 - z^{-1}\right)$

$\hookrightarrow z^{-1} = 1 : 3 - \frac{7}{2} = \frac{3}{2} A \Leftrightarrow \underline{A = -1/3}$

$\hookrightarrow z^{-1} = -2 : 3 + 7 = 3B \Leftrightarrow \underline{B = 10/3}$

So, in total:

$$H(z) = -2 - \frac{1}{3} \cdot \frac{1}{1 - z^{-1}} + \frac{10}{3} \frac{1}{1 + \frac{1}{2} z^{-1}}, \quad |z| < 1/2$$

z^{-1}

$$\boxed{h[n] = -2\delta[n] + \frac{1}{3} u[-n-1] - \frac{10}{3}\left(-\frac{1}{2}\right)^n u[-n-1]}$$

Exercise 2 (2016 final)

Causal and stable LTI system $H_1(z)$ is described by:

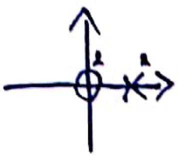
$$y[n] = x[n] + y[n-1] - \frac{1}{4} y[n-2], \quad (i)$$

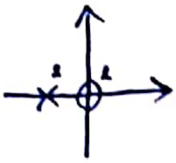
with $H_2(z) = H_1(-z)$, another system, (ii).

(a') Poles / zeroes diagram for both systems.

Ans

$Z\{(i)\} \Rightarrow Y(z)\left(1 - z^{-1} + \frac{1}{4} z^{-2}\right) = X(z) \Leftrightarrow$





$$(=)\ \boxed{H_1(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2}, \qquad H_2(z) = \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)^2}.}$$

so from (ii):

H_1: Double pole at $z=1/2$
 -//- zero at $z=0$

H_2: -//- pole at $z=-1/2$
 -//- zero at $z=0$

(β') For input $x[n] = A_1 e^{j\pi n/8} + A_2 e^{j(\pi-\pi/8)n}$, what is the percentage of amplitude change A_1, A_2 in the outputs of both systems?

Ans.

We know from theory the form of the output of an LTI system in case of exponential inputs (eigenvalues):

$$Y_1[n] = A_1 H_1(e^{j\pi/8})\, e^{j\pi n/8} + A_2 H_1\left(e^{j(\pi-\pi/8)}\right) e^{j(\pi-\pi/8)n}, \quad \text{or}$$

$$= A_1 \underbrace{|H_1(e^{j\pi/8})|}\, e^{j\varphi(\pi/8)} e^{j\pi n/8} + A_2 \underbrace{|H_1(e^{j(\pi-\pi/8)})|}\, e^{j\varphi(\pi-\pi/8)} \cdot e^{j(\pi-\pi/8)n}$$

we are only interested in these values for this question

- $$|H_1(e^{j\pi/8})| = \left|\frac{1}{\left(1-\frac{1}{2}e^{-j\pi/8}\right)^2}\right| = \dots = 1.42, \text{ (calculator)}$$

- $$|H_1(e^{j(\pi-\pi/8)})| = \left|\frac{1}{\left(1-\frac{1}{2}e^{-j(\pi-\pi/8)}\right)^2}\right| = \dots = 0.76, \text{ (-//-)}$$

Regarding $H_2(z)$, we will also get the same values, hence:

- A_1 increases by 42% for both systems, and
- A_2 decreases by 24% — " — .

(γ') Characterize these two systems as either lowpass, highpass, bandpass by justifying your answer.

Ans:

- $H_1(z)$ has its poles located in $\mathbb{R}^+$ positions => lowpass
- $H_2(z)$ —————— " —— $\mathbb{R}^-$ - " — => highpass

Exercise 3 (2016 final)

- A second order all-pass system has a pole at $z=3$, and a zero at $z=2$. Draw a graph that realizes this system using 2 multiplications and 3 delays (memory places). Multiplications with ± 1 are not counted.

Ans

↳ Second order all-pass : pole at $z=3 \Rightarrow$ zero at $z=1/3$

↳ — " — : zero at $z=2 \Rightarrow$ pole at $z=1/2$

• In total, it will be:

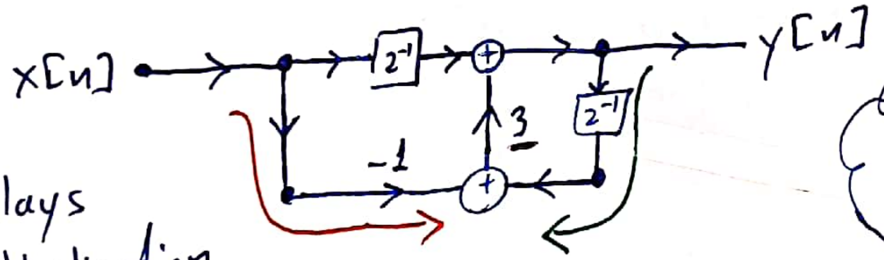
$$H_{ap}(z) = \frac{(z^{-1}-\frac{1}{2})(z^{-1}-3)}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})} = \overbrace{\frac{z^{-1}-\frac{1}{2}}{1-\frac{1}{2}z^{-1}}}^{H_1(z)} \cdot \overbrace{\frac{z^{-1}-3}{1-3z^{-1}}}^{H_2(z)} = \underbrace{H_1(z)H_2(z)}_{\text{also all-pass}}$$

• Drawing separately $H_1(z), H_2(z)$:

$$H_2(z) = \frac{z^{-1}-3}{1-3z^{-1}}$$

$\Leftrightarrow$

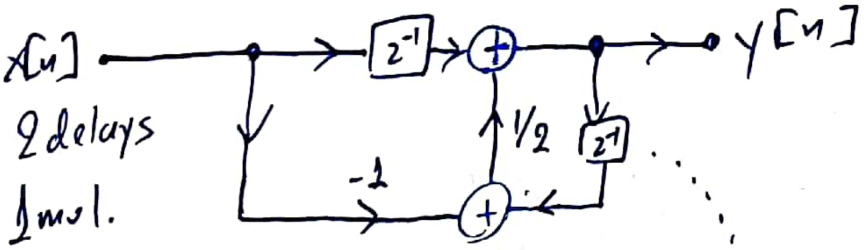
$$Y(z) = z^{-1}X(z) + \big((-1)X(z) + z^{-1}Y(z)\big)\cdot 3$$



2 delays
1 multiplication

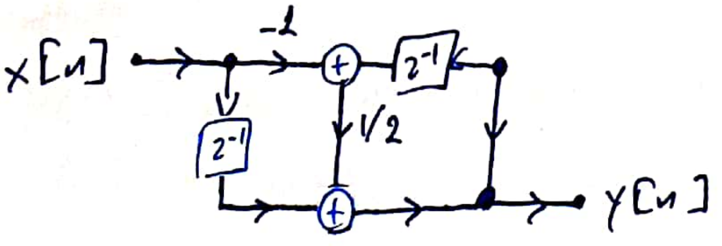
$$H_1(z) = \frac{z^{-1}-1/2}{1-\frac{1}{2}z^{-1}}$$

same, just with 1/2 instead of 3



2 delays
1 mul.

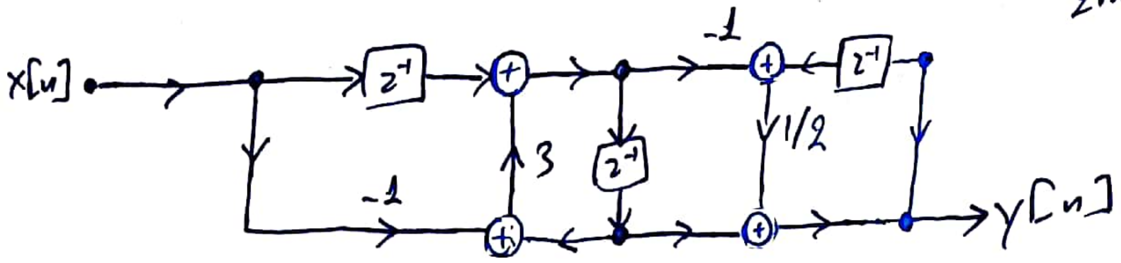
↳ we need 1 less delay, so we can make one be shared by both systems if we "flip" one of them like so:



(now we can connect it with $H_2(z)$ in series, or vise versa)

3 delays
2 multiplications

$H_{ap}(z)$.



Exercise 4 (2016 final)

Let $z_0 = 2e^{j2\pi/3}$, and the system:

$$H(z) = (1 - z_0 z^{-1})(1 - z_0^* z^{-1})\left(1 - \frac{1}{z_0} z^{-1}\right)\left(1 - \frac{1}{z_0^*} z^{-1}\right).$$

(a') Find a new system $H_1(z)$ such that: $|H_1(e^{j\omega})| = |H(e^{j\omega})|$.

Ans

We can find this system if we do all-pass-min-phase factorization, because it holds that:

does not affect amplitude

$$|H(e^{j\omega})| = |H_{min}(e^{j\omega})|\,|H_{ap}(e^{j\omega})| = A|H_{min}(e^{j\omega})| = |H_1(e^{j\omega})|$$

Hence, we essensially need to find the min-phase system.

- A min-phase system has all of its poles and zeros located within the unit circle, hence we will assign it the zeros at $1/z_0$ and $1/z_0^*$, while the other two zeros, z_0, z_0^* will go to the all-pass system. $H(z)$ does not have any other zeros, and all its poles are at $z=0$ (FIR = no poles).

- However, we know that all-pass systems have their poles and zeros in conjugate mutual pairs, so, since it has a zero at $1/z_0$, it must get a pole at z_0, and for the zero at $1/z_0^*$ it shall receive another pole at z_0^*:

$$H_{ap}(z) = \frac{(1-z_0 z^{-1})(1-z_0^* z^{-1})}{(1-\frac{1}{z_0}z^{-1})(1-\frac{1}{z_0^*}z^{-1})} =$$

converting it in the "all-pass form" (eq. 17.26a)

$$= \frac{(-z_0)(z^{-1}-\frac{1}{z_0})(-z_0^*)(z^{-1}-\frac{1}{z_0^*})}{(1-\frac{1}{z_0}z^{-1})(1-\frac{1}{z_0^*}z^{-1})} =$$

$$(-z_0)(-z_0^*) = (-2e^{j2\pi/3})(-2e^{-j2\pi/3}) = 4$$

$$= 4 \cdot \frac{(z^{-1}-\frac{1}{z_0})(z^{-1}-\frac{1}{z_0^*})}{(1-\frac{1}{z_0}z^{-1})(1-\frac{1}{z_0^*}z^{-1})}, \quad A = 4.$$

- These two additional poles we added to the all-pass system need to be cancelled-out by zeros in the corresponding places, otherwise we wouldn't get a correct factorization. Hence, the min-phase system gets two additional zeros at $1/z_0$, $1/z_0^*$, so the answer is:

$$H_{min}(z) = \left(1-\frac{1}{z_0}z^{-1}\right)^2\left(1-\frac{1}{z_0^*}z^{-1}\right)^2$$

$$H_1(z) = A\,H_{min}(z) = 4\left(1-\frac{1}{z_0}z^{-1}\right)^2\left(1-\frac{1}{z_0^*}z^{-1}\right)^2.$$

(β') Explain whether $H(z)$ is linear phase or not.

Ans

Since all of its zeros come in conjugate mutual positions $(z, 1/z, z^*, 1/z^*)$, this is a necessary and sufficient condition for linear phase systems. So, $H(z)$ is linear phase.

(γ') Explain if $H_1(z)$ is linear phase or not.

Ans

Its zeros are not in conjugate mutual positions, i.e., there is no zero at z_0 or z_0^*, so it cannot be a linear phase system.

(δ') Plot the impulse response $h[n]$.

Ans

$$H(z) = (1 - z_0 z^{-1})(1 - z_0^* z^{-1})\left(1 - \frac{1}{z_0} z^{-1}\right)\left(1 - \frac{1}{z_0^*} z^{-1}\right)$$

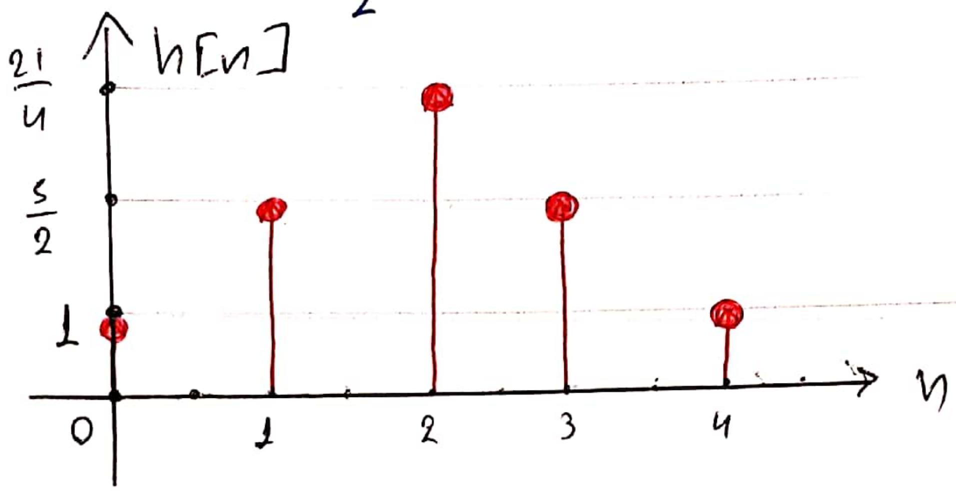
$$= \left(1 - \mathrm{Re}\{z_0\} z^{-1} + 4z^{-2}\right)\left(1 - 2\mathrm{Re}\{\tfrac{1}{z_0}\} z^{-1} + \frac{1}{4} z^{-2}\right)$$

$$= \left(1 + 2z^{-1} + 4z^{-2}\right)\left(1 + 2z^{-1} + \frac{1}{4} z^{-2}\right)$$

$$= 1 + \frac{5}{2} z^{-1} + \frac{21}{4} z^{-2} + \frac{5}{2} z^{-3} + z^{-4}$$

z^{-1}

$$h[n] = \delta[n] + \frac{5}{2}\delta[n-1] + \frac{21}{4}\delta[n-2] + \frac{5}{2}\delta[n-3] + \delta[n-4]$$



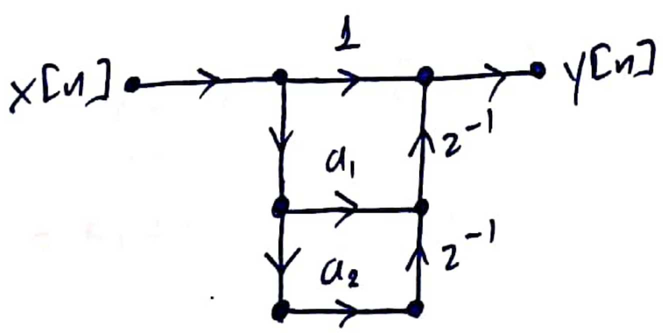
Exercise 5 (2016 Final)

Let a FIR system: $y[n] = x[n] + a_1 x[n-1] + a_2 x[n-2]$

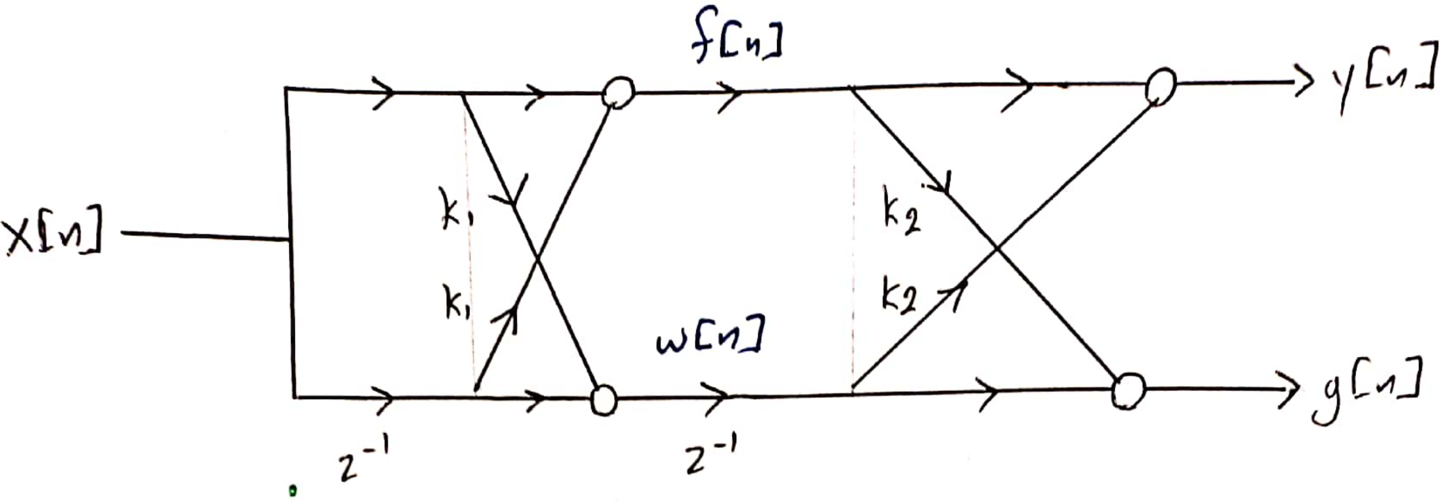
(a') Draw a classic realization of it.

Ans

Here is a simple one:



(b') What relationship connects a_1, a_2 with the k_1, k_2 coefficients of the following graph, such that $y[n]$ remains the same?



Ans

We set $f[n]$, $w[n]$ as intermediate values where:

$$f[n] = x[n] + k_1 x[n-1] \quad , (i)$$

$$w[n] = k_1 x[n] + x[n-1] \quad , (ii)$$

$$y[n] = f[n] + k_2 w[n-1] \overset{(i)}{\underset{(ii)}{\Longrightarrow}}$$

$$\Rightarrow y[n] = x[n] + k_1 x[n-1] + k_2 k_1 x[n-1] + k_2 x[n-2] \quad (=)$$

$$\left.\begin{aligned} y[n] &= x[n] + (k_1 + k_1 k_2)\, x[n-1] + k_2 x[n-2] \\ y[n] &= x[n] + a_1 x[n-1] + a_2 x[n-2] \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\begin{aligned} a_1 &= k_1 + k_1 k_2 = k_1(1+k_2) \quad , (iii) \\ a_2 &= k_2 . \quad , (iv) \end{aligned}}$$

(δ') Find the difference equation for the output $g[n]$ with respect to a_1, a_2.

Ans

From the graph we get:

$$g[n] = k_2 f[n] + w[n-1] \quad \overline{(ii)}^{(i)}$$

$$= k_2 x[n] + (k_1 + k_1 k_2) x[n-1] + x[n-2] \quad \overline{(iv)}^{(iii)}$$

$$= a_2 x[n] + a_1 x[n-1] + x[n-2], \quad (v).$$

(5') We can write y[n]'s difference equation as $Y(z) = A(z) X(z)$, in z domain. Write the z transform of g[n], i.e., G(z), as a function of A(z).

Ans

- $y[n] = x[n] + a_1 x[n-1] + a_2 x[n-2]$ $\swarrow z$

$Y(z) = X(z)(1 + a_1 z^{-1} + a_2 z^{-2})$, so

$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$.

- $z\{(v)\} \Rightarrow G(z) = X(z)(a_2 + a_1 z^{-1} + z^{-2})$, so since:

$z^{-2} A(z^{-1}) = z^{-2}(1 + a_1 z + a_2 z^2) = z^{-2} + a_1 z^{-1} + a_2$, it is

$$\boxed{G(z) = z^{-2} A(z^{-1}) X(z)}$$

(ε') Draw the poles and zeros of $G(z)$, given that:

$$A(z) = 1 - z^{-1} + \frac{1}{2} z^{-2}$$

Ans

It is mostly just about factorizing $A(z)$:

- $\frac{1}{2} z^{-2} - z^{-1} + 1$, $\Delta = 1 - 4\frac{1}{2} = -1$, $z_{1,2} = \frac{1 \pm j}{2} = \frac{\sqrt{2}}{2} e^{\pm j\pi/4}$

- $e^{jx} = \cos(x) + j\sin(x)$ =>
  $e^{\pm j\pi/4} = \frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$ <=>
  $\frac{\sqrt{2}}{2} e^{\pm j\pi/4} = \frac{1 \pm j}{2}$.

Let $z_0 = z_1$, so $z_0^* = z_2$.
Hence, we can factorize $A(z)$ like so:

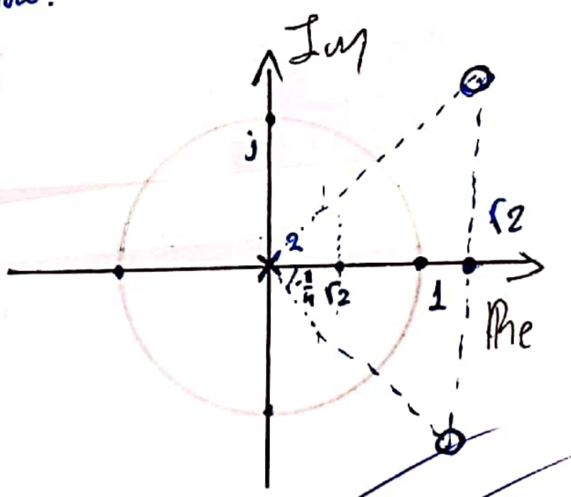
$$A(z) = (1 - z_0 z^{-1})(1 - z_0^* z^{-1})$$

we do not know!

$$G(z) = z^{-2} A(z^{-1}) X(z)$$

2 poles at zero and 2 zeros at infinity

2 zeros at $\sqrt{2} \cdot e^{\pm j\pi/4}$



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