HY-370 Fall 2022-23

7th Assistant Lecture

TA: George Manos csd4333

Exercise 1

Find the coefficients of the FIR linear phase system:

$$y[n] = b_0 \cdot x[n] + b_1 \cdot x[n-1] + b_2 \cdot x[n-2]$$

where $b_0, b_1, b_2 \neq 0$, so that

i) Completely cuts off frequency $\omega_0 = \frac{2\pi}{3}$

ii) $H(e^{j0}) = 1$

Also calculate the amplitude response and the phase response.

Solution

First, we need to calculate $Y(e^{j\omega})$.

$$Y(e^{j\omega}) = b_0 \cdot X(e^{j\omega}) + b_1 \cdot X(e^{j\omega}) \cdot e^{-j\omega} + b_2 \cdot X(e^{j\omega}) \cdot e^{-j2\omega}$$

$$= (b_0 + b_1 \cdot e^{-j\omega} + b_2 \cdot e^{-j2\omega}) \cdot X(e^{j\omega}) \Rightarrow$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = b_0 + b_1 \cdot e^{-j\omega} + b_2 \cdot e^{-j2\omega}$$

We know that $H(e^{j0}) = 1$, so $H(e^{j0}) = b_0 + b_1 + b_2 = 1$ (1)

from (i): $H(e^{j\frac{2\pi}{3}}) = 0 \Rightarrow b_0 + b_1 \cdot e^{-j\frac{2\pi}{3}} + b_2 \cdot e^{-j\frac{4\pi}{3}} = 0$

$\Leftrightarrow b_0 + b_1 \cdot \left(\cos\left(\frac{2\pi}{3}\right) - j\sin\left(\frac{2\pi}{3}\right)\right) + b_2\left(\cos\left(\frac{4\pi}{3}\right) - j\sin\left(\frac{4\pi}{3}\right)\right) = 0$

$\Leftrightarrow b_0 + b_1\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + b_2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 0$

$\Leftrightarrow \underline{\underline{2b_0}} - \underline{\underline{b_1}} - \underline{jb_1\sqrt{3}} - \underline{\underline{b_2}} + \underline{b_2 j\sqrt{3}} = 0$

$\Leftrightarrow \begin{matrix} R \\ Im \end{matrix}\begin{cases} 2b_0 - b_1 - b_2 = 0 \\ -\sqrt{3}b_1 + \sqrt{3}b_2 = 0 \end{cases} \Leftrightarrow \begin{cases} b_0 = b_1 = b_2 \\ b_1 = b_2 \end{cases} \overset{(1)}{\Leftrightarrow} b_0 = b_1 = b_2 = \frac{1}{3}$

So:

$$H(e^{j\omega}) = \frac{1}{3} + \frac{1}{3} \cdot e^{-j\omega} + \frac{1}{3} \cdot e^{-j2\omega} = \frac{1}{3} \cdot e^{-j\omega}(e^{j\omega} + 1 + e^{-j\omega})$$

$$= \frac{1}{3} \cdot e^{-j\omega} \cdot (2\cos(\omega) + 1)$$

The amplitude response:

$$|H(e^{j\omega})| = \frac{1}{3} \cdot 1 \cdot |2\cos(\omega) + 1|$$

and the phase reponse:

$$\measuredangle H(e^{j\omega}) = \measuredangle\frac{1}{3} + \measuredangle e^{-j\omega} + \measuredangle(2\cos(\omega) + 1)$$

$$= 0 - \omega + \underline{\measuredangle(2\cos(\omega) + 1)} \rightarrow = 0 \text{ for } \omega = \pm\frac{2\pi}{3}$$

different sign on either side of root

So:

$$\measuredangle(2\cos\omega+1)=\begin{cases}0, & \omega\in\left(-\frac{2\pi}{3},\frac{2\pi}{3}\right)\\ \pi, & \omega\in\left(\frac{2\pi}{3},\pi\right)\\ -\pi, & \omega\in\left(-\pi,-\frac{2\pi}{3}\right)\end{cases}\qquad \left(\begin{matrix}2\cos\omega+1<0\\ \omega>0\end{matrix}\right)$$

Finally:

$$\measuredangle H(e^{j\omega})=\begin{cases}-\omega, & \omega\in\left(-\frac{2\pi}{3},\frac{2\pi}{3}\right)\\ -\omega+\pi, & \omega\in\left(\frac{2\pi}{3},\pi\right)\\ -\omega-\pi, & \omega\in\left(-\pi,-\frac{2\pi}{3}\right)\end{cases}$$

Exercise 2

A linear phase filter:

$$H_1(z)=1-z^{-1}$$

(i) Factorize $H_1(e^{j\omega})$ as:

$$H_1(e^{j\omega})=A(e^{j\omega})\cdot e^{j\phi(e^{j\omega})}$$

where $A(e^{j\omega})\in\mathbb{R}$

(ii) The $H_1(z)$ system is connected in series with a generalized linear phase system $H_2(e^{j\omega})=A_2(e^{j\omega})\cdot e^{-j\omega\frac{M}{2}}$, type II.

Show that the overall system is linear phase of type III.

Solution

a) $H(e^{j\omega}) = 1 - e^{-j\omega} = e^{-j\frac{\omega}{2}} \cdot \left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right) = e^{-j\frac{\omega}{2}} \cdot j \cdot 2 \cdot \sin(\frac{\omega}{2})$

$= e^{-j\frac{\omega}{2}} \cdot e^{j\frac{\pi}{2}} \cdot 2 \cdot \sin(\frac{\omega}{2}) = e^{j(-\frac{\omega}{2}+\frac{\pi}{2})} 2 \cdot \sin(\frac{\omega}{2})$

So $H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{j\phi(e^{j\omega})}$

where $A(e^{j\omega}) = 2\sin(\frac{\omega}{2})$ and $\phi(e^{j\omega}) = -\frac{\omega}{2} + \frac{\pi}{2}$

So its an FIR type IV linear phase system

b) Since they are connected in series, the total impulse response is:

$h[n] = h_1[n] * h_2[n] \Leftrightarrow$

$H(e^{j\omega}) = H_1(e^{j\omega}) \cdot H_2(e^{j\omega}) = A_1(e^{j\omega}) \cdot A_2(e^{j\omega}) \cdot e^{j(\phi_1(e^{j\omega}) + \phi_2(e^{j\omega}))}$

$= \underbrace{2\sin(\frac{\omega}{2}) \cdot A_2(e^{j\omega})}_{A_3(e^{j\omega}) \in \mathbb{R}} \cdot e^{j\underbrace{(\phi_1(e^{j\omega}) + \phi_2(e^{j\omega}))}_{\frac{-\omega+\pi-\omega \cdot M}{2} = -\frac{\omega(M+1)}{2} + \frac{\pi}{2}}}$

Since $H_2(e^{j\omega})$ is type II, M is odd, so $\frac{M+1}{2}$ is an integer

So the total system $h[n]$ is a type III linear phase.

Exercise 3

Given the frequency response of an LTI system as:

$$H(e^{j\omega}) = e^{-j3\omega}\left(1+\cos(\omega)+\frac{2}{5}\cos(2\omega)-\frac{1}{5}\cos(3\omega)\right)$$

i) Classify the filter as FIR or IIR.

ii) Find the impulse response $h[n]$

iii) Draw a graph that implements the system.

Solution

i) Using Euler, the cosines are refactored to exponentials, which in turn correspond to delta functions in ~~frequency~~ time domain. So, the system is FIR.

ii) $$H(e^{j\omega}) = e^{-j3\omega}\left(1+\frac{e^{j\omega}+e^{-j\omega}}{2}+\frac{2}{5}\frac{e^{j2\omega}+e^{-j2\omega}}{2}-\frac{1}{5}\cdot\frac{e^{j3\omega}+e^{-j3\omega}}{2}\right)$$

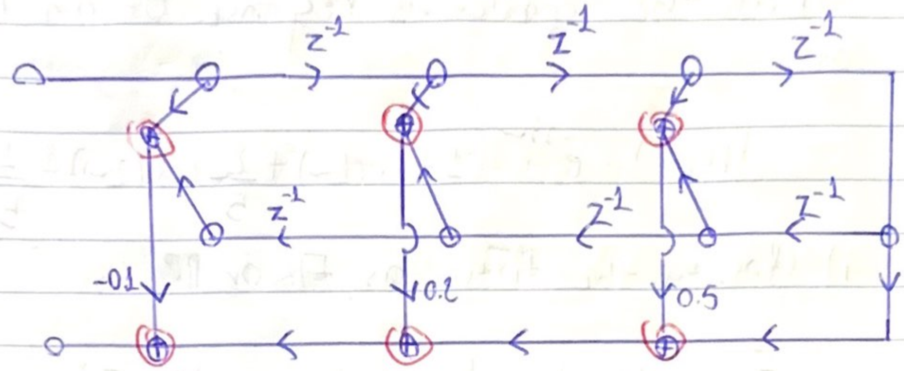
$$= e^{-j3\omega} - 0.1\cdot[1+e^{-j6\omega}] + 0.2\cdot[e^{-j\omega}+e^{-j5\omega}] + 0.5[e^{-j2\omega} + e^{-j4\omega}]$$

So,

$$h[n] = \delta[n-3] - 0.1\cdot\delta[n] - 0.1\cdot\delta[n-6] + 0.2\cdot\delta[n-1] + 0.2\cdot\delta[n-5] + 0.5\cdot\delta[n-2] + 0.5\cdot\delta[n-4] \Rightarrow$$

$$h[n] = -0.1\cdot\delta[n] + 0.2\cdot\delta[n-1] + 0.5\cdot\delta[n-2] + \delta[n-3] + 0.5\cdot\delta[n-4] + 0.2\cdot\delta[n-5] - 0.1\cdot\delta[n-6]$$

iii) Its a linear phase type I system.



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Exercise 4

Draw the graph of the following LTI System:

$$H(z) = \frac{16+9z^{-1}-z^{-2}}{1+\frac{1}{4}\cdot z^{-1}-\frac{1}{8}\cdot z^{-2}}$$

as a parallel form with first order subsystems in Direct Form.

Solution

We need to refactor before drawing in parallel form systems. The polynomial order of the dividend is equal to the order of the divisor, so we need to perform a polynomial division.

$$H(z) = 8 + \frac{8+7z^{-1}}{1+\frac{1}{4}\cdot z^{-1}-\frac{1}{8}\cdot z^{-2}} = 8 + \frac{8+7z^{-1}}{(1-\frac{1}{4}\cdot z^{-1})(1+\frac{1}{2}z^{-1})}$$

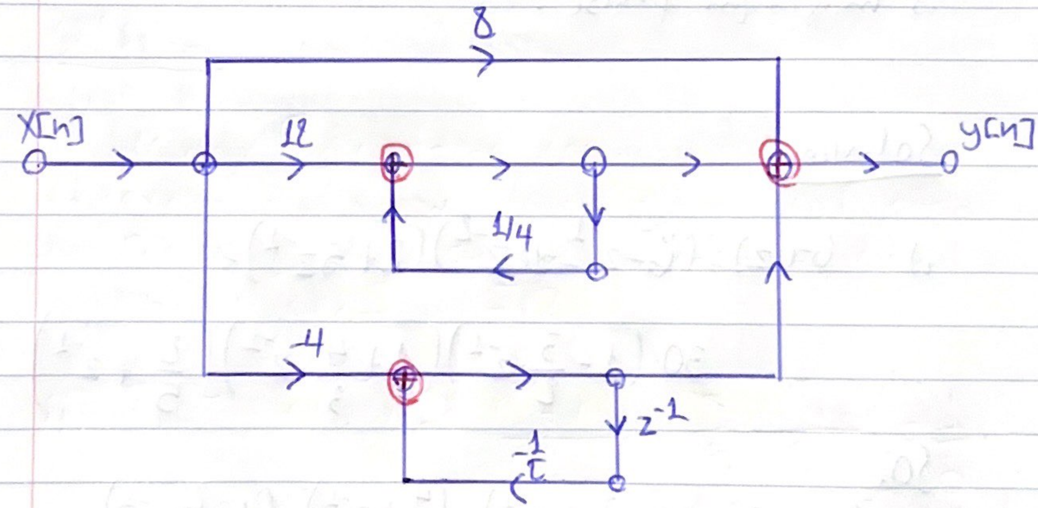
$$= 8 + \frac{A}{1-\frac{1}{4}\cdot z^{-1}} + \frac{B}{1+\frac{1}{2}\cdot z^{-1}}$$

Simple fractions decomposition: $A=12$, $B=-4$

So:

$$H(z) = 8 + \frac{12}{1-\frac{1}{4}\cdot z^{-1}} - \frac{4}{1+\frac{1}{2}\cdot z^{-1}}$$

Parallel form:



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Exercise 5

An FIR filter of third order has a transfer function $G(z)$

$$G(z) = (6 - z^{-1} - 12z^{-2})(2 + 5z^{-1})$$

i) Find all $H_i(z)$ of FIR filters so that

$$|H_i(e^{j\omega})| = |G(e^{j\omega})|$$

ii) Which one is a minimum phase and which is maximum phase?

Solution

i) $G(z) = (6 - z^{-1} - 12z^{-2})(2 + 5z^{-1}) =$

$$30 \cdot \left(1 - \frac{3}{2} \cdot z^{-1}\right)\left(1 + \frac{4}{3} \cdot z^{-1}\right)\left(\frac{2}{5} + z^{-1}\right)$$

So:

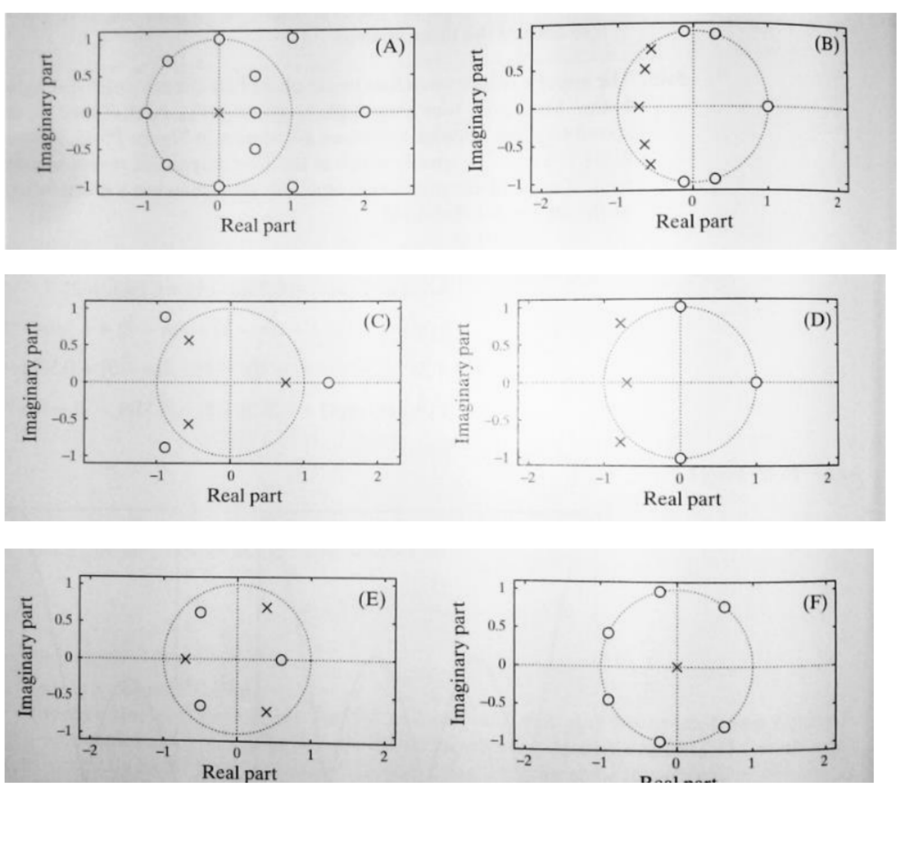
$$H_1(z) = 30\left(-\frac{3}{2} + z^{-1}\right) \cdot \left(\frac{4}{3} + z^{-1}\right) \cdot \left(1 + \frac{2}{5} \cdot z^{-1}\right)$$
$$H_2(z) = 30\left(-\frac{3}{2} + z^{-1}\right) \cdot \left(\frac{4}{3} + z^{-1}\right) \cdot \left(\frac{2}{5} + z^{-1}\right)$$
$$H_3(z) = 30\left(-\frac{3}{2} + z^{-1}\right)\left(1 + \frac{4}{3} z^{-1}\right) \cdot \left(1 + \frac{2}{5} \cdot z^{-1}\right)$$
$$H_4(z) = 30\left(-\frac{3}{2} + z^{-1}\right) \cdot \left(1 + \frac{4}{3} \cdot z^{-1}\right) \cdot \left(\frac{2}{5} + z^{-1}\right)$$
$$H_5(z) = 30\left(1 - \frac{3}{2} \cdot z^{-1}\right) \cdot \left(\frac{4}{3} + z^{-1}\right) \cdot \left(1 + \frac{2}{5} z^{-1}\right)$$
$$H_6(z) = 30\left(1 - \frac{3}{2} \cdot z^{-1}\right)\left(\frac{4}{3} + z^{-1}\right) \cdot \left(\frac{2}{5} + z^{-1}\right)$$
$$H_7(z) = 30\left(1 - \frac{3}{2} \cdot z^{-1}\right)\left(1 + \frac{4}{3} z^{-1}\right) \cdot \left(1 + \frac{2}{5} z^{-1}\right)$$
$$H_8(z) = G(z)$$

ii) $H_1(z)$ is the minimum phase and $H_8(z) = G(z)$ is the maximum phase

Exercise 6

The pole-zero plots of the following figure describe six different causal LTI systems. Show which of them are:

(α') IIR?
(β') FIR?
(γ') Stable?
(δ') Minimum phase?
(ε') Generalized Linear Phase?
(ϛ') $|H(e^{j\omega})| = a$, a constant $\forall \omega \in [-\pi, \pi]$?
(ζ') Stable and causal $H^{-1}(e^{j\omega})$ (inverse system)?
(η') minimum-duration-impulse response?
(θ') Spectral response with low-pass behavior?
(ι') Minimum group delay?



Solution:

α) B, C, D, E
β) A, F
γ) A, B, C, E, F
δ) E
ε) A, F
ϛ) C
ζ) E
η) F
θ) C, F
ι) E