$6^{\underline{o}}$ Φροντιστήριο .

Exercise 1

Let S_1 be a causal and stable LTI system with $h_1[n]$ and frequency response $H_1(e^{j\omega})$, and difference equation:

$$y[n] - y[n-1] + \frac{1}{4} y[n-2] = x[n] \quad ,(i)$$

i) If LTI S_2 has frequency response $H_2(e^{j\omega}) = H_1(-e^{j\omega})$, is it highpass, lowpass, or bandpass?

Ans:

$\cdot\ \mathcal{Z}\{(i)\} \Rightarrow Y(z)\left(1 - z^{-1} + \frac{1}{4} z^{-2}\right) = X(z) \quad (=)$

$$(=)\ H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{4} z^{-2}} = \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)^2} \quad , (ii)$$

$\cdot\ \underline{H_2(z) = H_1(-z)} = \dfrac{1}{\left(1 + \frac{1}{2} z^{-1}\right)^2} \rightarrow$ pole at $z^{-1} = -2 \Leftrightarrow z = -\frac{1}{2} = -\frac{1}{2} e^{-j\pi}$

Frequencies around $\omega = \pi$ are boosted, so this is a <u>highpass</u> filter.

(poles make the $\mathcal{Z}$ transform go to infinity there)

why can we do that? We know that same frequency response implies same location of poles/zeros, and their ROC will also be the same as they are both stable systems.

ii) Let S_3 be a causal LTI with the property:

$H_3(e^{j\omega})\, H_1(e^{j\omega}) = 1$, (iii). Is it minimum phase?

Is it among the four types of linear phase systems that you know of. Justify your answers.

Ans

(iii) tells us that S_3 is the inverse of S_1, i.e.,

$$H_3(z) = \frac{1}{H_1(z)} = \left(1 - \frac{1}{2} z^{-1}\right)^2 .$$

So $H_3(z)$ has a 2-nd rank zero at $z = \frac{1}{2} e^{j0}$, hindering frequencies around $\omega = 0$, thus, making it a <u>highpass</u> filter. Since its zeros are in the unit circle it is also <u>minimum phase</u>, but these zeros are not in conjugate mutual pairs making it <u>not linear phase</u>. We can also see the latter from:

$$\mathcal{Z}^{-1}\{H_3(z)\} \Rightarrow h_3[n] = \delta[n] - \delta[n-1] + \frac{1}{4}\delta[n-2],$$

where no characteristic symmetry of FIR linear phase systems is shown.

Exercise 2

We have 3 causal and real LTI systems with transfer functions $H_1(z)$, $H_2(z)$ and $H_3(z)$ respectively. Find as much information about them as you can regarding (a) their poles and zeros and (b) the duration of their impulse responses, if:

i. $H_1(z)$ has a pole at $z = 0.9e^{j\pi/3}$ and $x[n] = u[n] \rightarrow \lim_{n \to +\infty} y[n] = 0$.

Ans:

(a) real $\rightarrow$ conjugate(z, z^*) pairs of poles and zeros, or real poles and zeros.

causal $\rightarrow$ no poles at infinity

• So there is another pole at $z = 0.9e^{-j\pi/3}$ for $H_1(z)$.

• When $x[n] = u[n]$, $u[n] \xrightarrow{z} \frac{1}{1-z^{-1}}$, $|z| > 1$, then:

$Y(z) = H_1(z)X(z) = \dfrac{H_1(z)}{1-z^{-1}}$, which indicates that $H_1(z)$ must have a zero at $z = 1$. One way to think about, is that if we perform partial fraction decomposition (PFD) for $H_1(z)/1-z^{-1}$, then we will get a term like $A/1-z^{-1}$, which in time domain will be $Au[n]$, but at $n \to +\infty$ this won't fade to zero; it will remain constant (A). Hence, to cancel this term, there must be a $1-z^{-1}$ in the numerator, i.e, a zero at $z=1$.

(b) Since there is at least one pole for $H_1(z)$ (that is not cancelled out), then it must be of infinite duration (Z is infinite at poles etc.). In total:

$$H_1(z) = \frac{1-z^{-1}}{(1-0.9e^{j\pi/3}z^{-1})(1-0.9e^{-j\pi/3}z^{-1})} H_0(z),$$

with $H_0(z)$ keeping the rest of the information that we do not know for this system.

ii. $H_2(z)$ has a zero at $z = 0.8e^{j\pi/4}$, linear phase with group delay $grd\{H_2(e^{j\omega})\} = -2.5$ and $|H_2(e^{j0})| = 0$.

Ans:

(a)

↳ real, so zeros come in mutual conjugates $(z, z^*, \frac{1}{z}, \frac{1}{z^*})$ or $(0.8e^{j\pi/4}, 0.8e^{-j\pi/4}, 1.25e^{j\pi/4}, 1.25e^{-j\pi/4})$, in our case.

↳ causal + linear phase => has to be one of the four types.

$grd\{H(e^{j\omega})\} = -2.5 = -\frac{5}{2}$, so $\varphi(e^{j\omega}) = -\omega\frac{5}{2} + C$, with $M = 5$ in our case; and an odd M means it is either type II or IV.

That means that all poles of this system are at $z = 0$.

↳ $|H_2(e^{j0})| = 0$, so at $\omega = 0$, or $z = 1$, there is a zero.

(b) Since $M = 5$ then the duration is 6 samples in total. So:

$$H_2(z) = A(1 - z^{-1})(1 - z_1)(1 - z_1^*)(1 - 1/z_1)(1 - 1/z_1^*),$$

$z_1 = 0.8e^{j\pi/4}$, almost fully describing this system.

iii. $H_3(z)$ has a pole at $z = 0.8e^{j\pi/4}$, and $|H_3(e^{j\omega})| = 1, \forall\omega$.

Ans

(a)

↳ $|H_3(e^{j\omega})| = 1, \forall\omega$ => this is an all-pass system

↳ poles & zeros are in conjugate mutual pairs

(b) Infinite duration since poles exist. In total it will be:

$$H_3(z) = \frac{(z^{-1} - 0.8e^{j\pi/4})(z^{-1} - 0.8e^{-j\pi/4})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} \cdot H_{ap}(z),$$

→ an all-pass system with the rest of the poles and zeros we do not know about.

Exercise 3

LTI systems $H_1(e^{j\omega})$, $H_2(e^{j\omega})$ have generalized linear phase.
Which of the following systems are also linear phase?

i. $G_1(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$

Ans:

This is not necessarily linear phase because it is:

$$\measuredangle G_1(e^{j\omega}) = \tan^{-1} \frac{Im\{H_1(e^{j\omega}) + H_2(e^{j\omega})\}}{Re\{H_1(e^{j\omega}) + H_2(e^{j\omega})\}},$$

not always linear. As an example we can just use:

$$h_1[n] = \delta[n] + \delta[n-1]$$
$$h_2[n] = 2\delta[n] - 2\delta[n-1]$$
$$g_1[n] = h_1[n] + h_2[n] = 3\delta[n] - \delta[n-1] \quad \xrightarrow{F}$$
$$G_1(e^{j\omega}) = 3 - \cos(\omega) + j\sin(\omega)$$

$$\measuredangle G_1(e^{j\omega}) = \tan^{-1} \frac{\sin(\omega)}{3-\cos(\omega)},$$ not linear phase but

$H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are linear phase.

ii. $G_2(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$

Ans:

$\measuredangle G_2(e^{j\omega}) = \measuredangle H_1(e^{j\omega}) + \measuredangle H_2(e^{j\omega})$, summing two lines always gives another line. Hence, this system is * linear phase.

* always generalized

iii. $G_3(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\theta})\, H_2(e^{j(\omega-\theta)})\, d\theta.$

We can use a similar example to show that this is not* a linear phase system. We know that convolution in the Fourier domain translates to multiplication in time domain, and $G_3(e^{j\omega})$ here is given essentially by a convolution**! An example:

Both linear phase

$$h_1[n] = \delta[n] + \delta[n-1],$$

$$h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2],$$

$$g_3[n] = h_1[n] h_2[n] = \delta[n] + 2\delta[n-1]$$

$$\downarrow \mathcal{F}$$

$$G_3(e^{j\omega}) = 1 + 2\cos(\omega) - 2j\sin(\omega),$$

$$\measuredangle G_3(e^{j\omega}) = \tan^{-1}\left(\frac{2\sin(\omega)}{1+2\cos(\omega)}\right),$$ not linear phase.

*necessarily

Side note: • $\delta[n-n_0] \xrightarrow{\mathcal{F}} e^{-j\omega n_0} = \cos(\omega n_0) - j\sin(\omega n_0),$

from $e^{jx} = \cos(x) + j\sin(x).$

** • $x[n]y[n] \xrightarrow{\mathcal{F}} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\, d\theta.$

• $f[n]\delta[n-a] = f[a]\delta[n-a]$

Exercise 4 (from 2016 final exam)

The Fibonacci sequence (0,1,1,2,3,5,8,...) can be modelled as the impulse responce of the following system:

$$y[n] - y[n-1] - y[n-2] = x[n-1] \quad , (i)$$

Calculate this impulse response $h[n]$.

Ans

$$\mathcal{Z}\{(i)\} \Rightarrow Y(z)\left(1 - z^{-1} - z^{-2}\right) = X(z)\, z^{-1} \quad \Leftrightarrow$$

$$\Leftrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \quad , (ii), \text{ with}$$

poles at $1 - z^{-1} - z^{-2} = 0 \Leftrightarrow z_1, z_2 = \frac{1 \pm \sqrt{5}}{2}$, (iii).

(ii) PFD: $$\frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{A}{1 - z_1 z^{-1}} + \frac{B}{1 - z_2 z^{-1}} \quad \Leftrightarrow$$

$$\Leftrightarrow z^{-1} = A\left(1 - z_2 z^{-1}\right) + B\left(1 - z_1 z^{-1}\right)$$

$\hookrightarrow z^{-1} = z_2^{-1}$: $\frac{1}{z_2} = B\left(1 - \frac{z_1}{z_2}\right) \Leftrightarrow B = \frac{1}{z_2 - z_1} \overset{(iii)}{\Rightarrow} B = -\frac{1}{\sqrt{5}}$

$\hookrightarrow z^{-1} = z_1^{-1}$: ... $A = \frac{1}{\sqrt{5}}$

So in total:

$$H(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}z^{-1}}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1-\sqrt{5}}{2}z^{-1}}\right) \quad ,(iv)$$

- We know this system is <u>not stable</u> (observe that the Fibonacci sequence increases indefinetly).
- We know this system is <u>causal</u> (from (i) observe that there is no dependency on future values - only present/past).

$$Z^{-1}\{(iv)\} \Rightarrow \boxed{h[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].}$$

Exercise 5 (from 2016 final exam)

The real part of the Fourier transform of a real and causal signal $x[n]$ is $X_R(e^{j\omega}) = 2 - 4\cos(3\omega)$. Find $X(e^{j0})$ and $X(e^{j\pi})$.

Ans

- From theory (see table 13.4), we know that for a real $x[n]$, the real part of its Fourier transform corresponds to its even part, i.e. $x_e[n] \xleftrightarrow{F} X_R(e^{j\omega})$, which we can find:

$$X_R(e^{j\omega}) = 2 - 4\cos(3\omega) \Leftrightarrow$$

$$X_R(e^{j\omega}) = 2 - 2e^{j3\omega} - 2e^{-j3\omega}$$

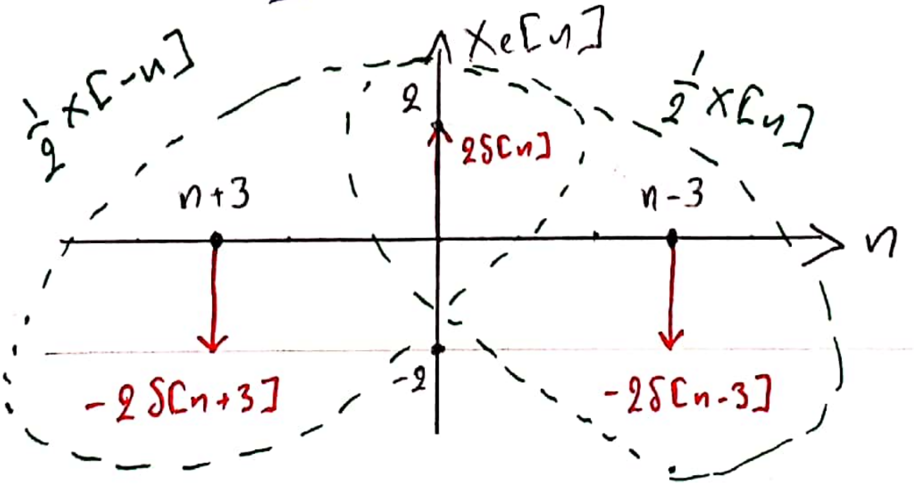
Eüler

F^{-1}

$$\underline{x_e[n] = 2\delta[n] - 2\delta[n+3] - 2\delta[n-3]}$$

- Even part means: $x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n] \Leftrightarrow$

$$\Leftrightarrow \underline{x[n] = 2x_e[n] - x[-n]}$$



↳ So, $x[n]$ will be:

$$x[n] = 2x_e[n] - x[-n] =$$

$$= 2\left(2\delta[n] - 2\delta[n+3] - 2\delta[n-3]\right) - \left(-2\delta[n+3] + 2\delta[n]\right)$$

but we know from the exercise that it is also causal, so it must be zero for all $n<0$ ($\delta[n+3]$ are discarded)

$$= 4\delta[n] - 4\delta[n-3] - 2\delta[n] = \underline{2\delta[n] - 4\delta[n-3]} = x[n]$$

• Now we can easily calculate what's been asked:

$$\cdot\ X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{j\omega n} \Big|_{\omega=0} = \sum_{n=-\infty}^{+\infty} x[n] = 2-4 = -2.$$

$$\cdot\ X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{j\omega n} \Big|_{\omega=\pi} = \sum_{n=-\infty}^{+\infty} (-1)^n x[n] = 2+4 = 6.$$

ΤΕΛΟΣ