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CS370
5th Assistant Lecture
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Exercise 1

Assume an LTI system, described by the following transfer function:

$$H(z) = \frac{1+z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1+3z^{-1})}, \; R_H$$

Find the impulse response for each possible region of convergence (ROC), draw the corresponding poles and zeroes and comment on its stability and causality.

Solution:

$$H(z) = \frac{z^3+z^2}{(z-1)(z-\frac{1}{2})(z+3)} = \frac{z^2(z+1)}{(z-1)(z-\frac{1}{2})(z+3)}$$

Zeroes: $z=0$ (double), $z=-1$

poles: $z=1$, $z=\frac{1}{2}$, $z=-3$

ROC:

a) $|z|<\frac{1}{2}$, b) $\frac{1}{2}<|z|<1$, c) $1<|z|<3$

d) $|z| > 3$

More refactoring $H(z)$:

$$H(z) = \frac{A}{1 - z^{-1}} + \frac{B}{1 - \frac{1}{2} \cdot z^{-1}} + \frac{C}{1 + 3z^{-1}}$$

(Partial fractions decomposition)

Where: $A = 1$, $B = -\frac{2}{7}$, $C = \frac{3}{7}$

Therefore:

$$H(z) = \frac{1}{1 - z^{-1}} - \frac{2}{7} \frac{1}{1 - \frac{1}{2} \cdot z^{-1}} + \frac{3}{7} \cdot \frac{1}{1 + 3z^{-1}}$$

Case a):

$$h[n] = -u[-n-1] + \frac{2}{7} \cdot \left(\frac{1}{2}\right)^n \cdot u[-n-1] - \frac{3}{7} \cdot (-3)^n \cdot u[-n-1]$$

Case b):

$$h[n] = -u[-n-1] - \frac{2}{7} \cdot \left(\frac{1}{2}\right)^n \cdot u[n] - \frac{3}{7} \cdot (-3)^n \cdot u[-n-1]$$

Case c)

$$h[n] = u[n] - \frac{2}{7}\left(\frac{1}{2}\right)^n \cdot u[n] - \frac{3}{7} \cdot (-3)^n \cdot u[-n-1]$$

Case d):

$$h[n] = u[n] - \frac{2}{7} \cdot \left(\frac{1}{2}\right)^n \cdot u[n] + \frac{3}{7} \cdot (-3)^n \cdot u[n]$$

	Stable	Causal
a)	X	X
b)	X	X
c)	X	X
d)	X	✓

Exercise 2

Consider the following system functions for stable LTI systems. Without utilizing the inverse z-transform, determine in each case whether or not the corresponding system is causal

a) $$\frac{1-\frac{4}{3}\cdot z^{-1}+\frac{1}{2}\cdot z^{-2}}{z^{-1}\cdot\left(1-\frac{1}{2}\cdot z^{-2}\right)\left(1-\frac{1}{3}\cdot z^{-1}\right)} = \frac{z^3-\frac{4}{3}\cdot z^2+\frac{1}{2}\cdot z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

It has a pole in infinity, so it's not causal.

b) $$\frac{z-\frac{1}{2}}{z^2+\frac{1}{2}\cdot z-\frac{3}{16}} = \frac{z-\frac{1}{2}}{\left(z+\frac{3}{4}\right)\left(z-\frac{1}{4}\right)}$$

poles in $z=\frac{1}{4}, -\frac{3}{4}$

So it is causal for $|z|>\frac{3}{4}$

c) $$\frac{z+1}{z+\frac{4}{3}-\frac{1}{2}\cdot z^{-2}-\frac{2}{3}\cdot z^{-3}} = \frac{z+1}{\left(z+\frac{4}{3}\right)\left(1-\frac{1}{2}z^{-3}\right)}$$

$$= \frac{z^4+z^3}{\left(z+\frac{4}{3}\right)\left(z^3-\frac{1}{2}\right)},$$

poles in $z=-\frac{4}{3}$, $\sqrt[3]{\frac{1}{2}}$

Non-causal!

Exercise 3

Consider a linear phase discrete-time LTI system with frequency response $H(e^{j\omega})$ and real impulse response $h[n]$. The group delay function for such a system is defined as:

$$\tau(\omega) = -\frac{\partial}{\partial \omega} \measuredangle H(e^{j\omega}),$$

where $\measuredangle H(e^{j\omega})$ has no discontinuites. Suppose that, for this system:

$$|H(e^{j\frac{\pi}{2}})| = 2 \quad \text{and} \quad \tau\left(\frac{\pi}{2}\right) = 2$$

Determine the output of the system for each of the following inputs:

(a) $\cos\left(\frac{\pi}{2} \cdot n\right)$ (b) $\sin\left(\frac{7\pi}{2} \cdot n + \frac{\pi}{4}\right)$

(a) $\cos\left(\frac{\pi}{2} \cdot n\right) = x_1[n] + x_2[n]$, where

$$x_1[n] = \frac{1}{2} \cdot e^{j\pi \frac{n}{2}}, \quad x_2[n] = x_1^*[n]$$

When $x_1[n]$ passes through the system, a delay of 2 samples is applied to it.

Because $h[n]$ is real, then the group delay is an even parity function:

$$y[n] = 2 \cdot x_1[n-2] + 2x_2[n-2] = 2\cos\left(\frac{\pi}{2}(n-2)\right)$$

n integer!

(b) $x[n] = \sin\left(\frac{7\pi n}{2} + \frac{\pi}{4}\right) = \sin\left(\cancel{4\pi n} - \frac{\pi n}{2} + \frac{\pi}{4}\right)$

$- \sin\left(\frac{\pi}{2} n - \frac{\pi}{4}\right)$

Same as above, we can refactor the expression to a sum of exponentials with frequencies

$\omega = \frac{\pi}{2}, -\frac{\pi}{2}$

The output is expressed as:

$$y[n] = 2x[n-2] = 2\sin\left(\frac{7\pi}{2}(n-2) + \frac{\pi}{4}\right) 2\sin\left(\frac{7\pi}{2} n - \frac{3\pi}{4}\right)$$

Exercise 4

Consider a causal LTI System whose frequency response is given as:

$$H(e^{j\omega}) = e^{-j\omega} \cdot \frac{1 - \frac{1}{2} \cdot e^{j\omega}}{1 - \frac{1}{2} \cdot e^{-j\omega}}$$

(a) Show that $|H(e^{j\omega})|$ is unity at all frequencies

(b) Show that

$$\measuredangle H(e^{j\omega}) = -\omega - 2\tan^{-1}\left(\frac{\frac{1}{2} \cdot \sin\omega}{1 - \frac{1}{2} \cdot \cos\omega}\right)$$

(C) Show that the group delay for this filter is given by

$$\tau(\omega) = \frac{\frac{3}{4}}{\frac{5}{4} - \cos\omega}$$

Sketch $\tau(\omega)$

(d) What is the output of this filter when the input is $\cos(\frac{\pi}{3} n)$?

Solution

(a) $$|H(e^{j\omega})| = \left| e^{-j\omega} \cdot \frac{1 - \frac{1}{2} \cdot e^{j\omega}}{1 - \frac{1}{2} \cdot e^{-j\omega}} \right| = |e^{-j\omega}| \frac{|1 - \frac{1}{2} \cdot e^{j\omega}|}{|1 - \frac{1}{2} \cdot e^{-j\omega}|}$$

$= 1$

(b) $$\measuredangle H(e^{j\omega}) = \measuredangle e^{-j\omega} + \measuredangle\left(1 - \frac{1}{2} \cdot e^{j\omega}\right) - \measuredangle\left(1 - \frac{1}{2} \cdot e^{-j\omega}\right)$$

$$= \measuredangle e^{-j\omega} +$$

$$\measuredangle\left(1 - \frac{1}{2} \cdot \cos(\omega) - \frac{j}{2} \sin(\omega)\right) -$$

$$\measuredangle\left(1 - \frac{1}{2} \cos(\omega) + \frac{j}{2} \sin(\omega)\right) = -\omega - 2\tan^{-1}\left(\frac{\frac{1}{2} \cdot \sin(\omega)}{1 - \frac{1}{2} \cos(\omega)}\right)$$

(c) $\tau(\omega) = \dfrac{-\partial \measuredangle H(e^{j\omega})}{\partial \omega} = 1 + 2\, \dfrac{1}{\underbrace{1+\left(\dfrac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right)^2}_{\dfrac{(1-\frac{1}{2}\cos(\omega))^2}{1-\cos(\omega)+\frac{1}{4}}}} \cdot$

$$\underbrace{\frac{\frac{1}{2}\cos(\omega)(1-\frac{1}{2}\cos(\omega)) - \frac{1}{4}\sin^2(\omega)}{(1-\frac{1}{2}\cos(\omega))^2}}$$

$$\downarrow$$

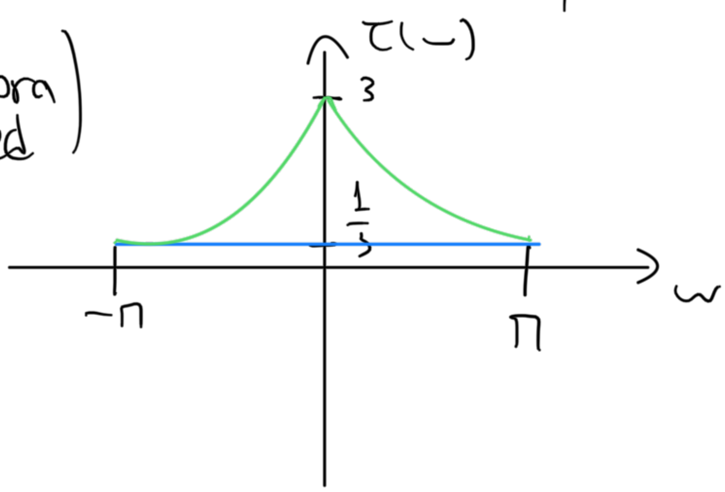
$$\frac{\frac{1}{2}\cos(\omega) - \frac{1}{4}\cos^2(\omega) - \frac{1}{4}\sin^2(\omega)}{(1-\frac{1}{2}\cos(\omega))^2} = \frac{\frac{1}{2}\cos(\omega) - \frac{1}{4}}{(1-\frac{1}{2}\cos(\omega))^2}$$

So:

$$\tau(\omega) = 1 + 2\,\frac{\cancel{(1-\frac{1}{2}\cos(\omega))^2}}{1-\cos(\omega)+\frac{1}{4}} \cdot \frac{\frac{1}{2}\cos(\omega) - \frac{1}{4}}{\cancel{(1-\frac{1}{2}\cos(\omega))^2}}$$

$$= \frac{\frac{5}{4} - \cancel{\cos(\omega)} + \cancel{\cos(\omega)} - \frac{1}{2}}{\frac{5}{4} - \cos(\omega)} = \frac{\frac{3}{4}}{\frac{5}{4} - \cos(\omega)}$$

Sketch: (geogebra required)



(d) $x[n] = \cos\left(\frac{\pi}{3} \cdot n\right) = \frac{e^{j\frac{\pi}{3}n}}{2} + \frac{e^{-j\frac{\pi}{3}n}}{2}$

From (c):

$$\tau\left(\frac{\pi}{3}\right) = 1 = \tau\left(-\frac{\pi}{3}\right)$$

So, the output is: $y[n] = \frac{e^{j\pi\frac{(n-1)}{3}}}{2} + \frac{e^{-j\pi\frac{(n-1)}{3}}}{2}$

$$= \cos\left(\frac{(n-1)\cdot\pi}{3}\right)$$

Exercise 5

Consider the rectangular signal

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{Otherwise} \end{cases}$$

Let $g[n] = x[n] - x[n-1]$

(a) find the signal $g[n]$ and directly evaluate its z-transform.

(b) Noting that

$$x[n] = \sum_{k=-\infty}^{n} g[k]$$

Determine the z-transform of $x[n]$

Solution

(a) $g[n] = \delta[n] - \delta[n-6]$

$$\downarrow \mathcal{Z}\{\}$$

$$G(z) = 1 - z^{-6}, \quad |z| > 0$$

(b) $x[n] = \sum_{k=-\infty}^{n} g[k]$

$$\updownarrow \mathcal{Z}()$$

$$X(z) = \frac{1}{1-z^{-1}} \cdot G(z) = \frac{1-z^{-6}}{1-z^{-1}} = \frac{z^6-1}{z^6-z^5}$$

Pole in $z=0$. $|z|>0$

$x[n]$ is a finite length signal, so ROC $|z|>0$