CS 370 - Signal Processing
2nd Recitation
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HW2 - 2019

Exercise 4

Using Only properties and pairs of Fourier Transform, prove that

$$\left(\frac{1}{4}\right)^{|n|} \underset{F^{-1}\{\}}{\overset{F\{\}}{\rightleftarrows}} \frac{\frac{15}{16}}{\frac{17}{16} - \frac{1}{2}\cos(\omega)}$$

Solution

The signal may be expressed as:

$$x[n] = \left(\frac{1}{4}\right)^{|n|} = \left(\frac{1}{4}\right)^{n} \cdot u[n] + \left(\frac{1}{4}\right)^{-n} \cdot u[-n] - \delta[n]$$

From the pairs table:

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}\cdot e^{-j\omega}} + \frac{1}{1-\frac{1}{4}\cdot e^{j\omega}} - 1$$

$$= \frac{1-\frac{1}{4}\cdot e^{j\omega} + 1-\frac{1}{4}\cdot e^{-j\omega}}{\left(1-\frac{1}{4}\cdot e^{-j\omega}\right)\left(1-\frac{1}{4}\cdot e^{j\omega}\right)} - 1 = \frac{2-\frac{1}{4}\left(e^{j\omega}+e^{-j\omega}\right)}{\left(1-\frac{1}{4}\cdot e^{-j\omega}\right)\left(1-\frac{1}{4}\cdot e^{j\omega}\right)} - 1$$

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{|1 - \frac{1}{4}\cdot e^{-j\omega}|^2} - 1$$ (Conjugate property, Euler)

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{(1 - \frac{1}{4}\cdot\cos(\omega))^2 + (\frac{1}{4}\sin(\omega))^2} - 1$$ $\begin{pmatrix} \text{Euler, } e^{-j\varphi} = \cos\varphi - j\sin\varphi \\ \text{complex, } |z|^2 = Re(z)^2 + Im(z)^2 \end{pmatrix}$

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{\left(\frac{17}{16} - \frac{1}{2}\cos(\omega)\right)} - 1 = \frac{2 - \frac{1}{2}\cos(\omega)}{\left(\frac{17}{16} - \frac{1}{2}\cos(\omega)\right)} - \frac{\left(\frac{17}{16} - \frac{1}{2}\cos(\omega)\right)}{\left(\frac{17}{16} - \frac{1}{2}\cos(\omega)\right)}$$

$$= \frac{\frac{15}{16}}{\frac{17}{16} - \frac{1}{2}\cos(\omega)}$$

Exercise 5

Assume input and Impulse Response to a system:

$$x[n] = -3^n \cdot u[-n-1], \quad h[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$$

Calculate the expression for the system output $y[n]$ using properties and pairs for Fourier Transform:

Solution

Using pairs, we have:

$$X(e^{j\omega}) = \frac{1}{1 - 3 \cdot e^{j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} \cdot e^{j\omega}}$$

So

$$Y(e^{j\omega}) = X(e^{j\omega}) \, H(e^{j\omega}) = \frac{1}{1 - 3 \cdot e^{j\omega}} \; \frac{1}{1 - \frac{1}{2} \cdot e^{j\omega}}$$

$$= \frac{A}{1 - 3 \cdot e^{-j\omega}} + \frac{B}{1 - \frac{1}{2} \cdot e^{-j\omega}}$$

(partial fractions decomposition)

where:

$$A = \left. \frac{1}{1 - \frac{1}{2} \cdot e^{-j\omega}} \right|_{e^{-j\omega} = \frac{1}{3}} = \frac{6}{5}$$

$$B = \left. \frac{1}{1 - 3e^{-j\omega}} \right|_{e^{-j\omega} = 2} = -\frac{1}{5}$$

So:

$$Y(e^{j\omega}) = \frac{\frac{6}{5}}{1 - 3e^{-j\omega}} - \frac{\frac{1}{5}}{1 - \frac{1}{2} \cdot e^{-j\omega}}$$

Using pairs table:

$$y[n] = -\frac{6}{5}\cdot 3^n u[-n-1] - \frac{2}{5}\left(\frac{1}{2}\right)^n \cdot u[n]$$

HW2-2018

Exercise 1

The input to an LTI system is expressed as:

$$x[n] = 4\cos\left(\frac{\pi n}{6} + \frac{\pi}{8}\right) - \sin\left(\frac{\pi n}{4} - \frac{\pi}{2}\right)$$

(a) Find the output of the system, $y[n]$, if the impulse response is expressed as:

$$h[n] = \frac{\sin\left(\frac{(n-2)\pi}{2}\right)}{(n-2)\pi}$$

(b) Compute the sum:

$$\sum_{n=-\infty}^{+\infty} \left| \frac{\sin(0.25\pi n)}{3\pi n} \right|^2$$

Solution

We denote that
$$\frac{\sin(\pi \frac{n}{2})}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow} \begin{cases} 1, & |\omega| \le \frac{\pi}{2} \\ 0, & \text{else} \end{cases}$$

So for the given impulse response:

$$h[n] = \frac{\sin\left(\frac{(n-2)\pi}{2}\right)}{(n-2)\pi} \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & |\omega| \le \frac{\pi}{2} \\ 0, & \text{else} \end{cases}$$

from using the time shift property.

This is a low-pass filter that discards frequencies $\notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, with fixed amplitude and phase -2ω. The input contains frequencies $\pm\frac{\pi}{6}, \pm\frac{\pi}{4}$, so both cosines will pass:

$$y[n] = 4\cos\left(\frac{\pi n}{6} + \frac{\pi}{8} + \angle H(e^{j\frac{\pi}{6}})\right) - \sin\left(\frac{\pi n}{4} - \frac{\pi}{2} + \angle H(e^{j\frac{\pi}{4}})\right)$$

$$= 4\cos\left(\frac{\pi n}{6} + \frac{\pi}{8} - \frac{\pi}{3}\right) - \sin\left(\frac{\pi n}{4} - \frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$= 4\cos\left(\frac{\pi n}{6} - \frac{5\pi}{24}\right) + \sin\left(\frac{\pi n}{4}\right)$$

(b) Remember Parseval's theorem?

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

Let $x[n] = \frac{1}{3}\frac{\sin(\pi\frac{n}{4})}{\pi n}$.

X corresponds to a low pass filter with fixed amplitude $= \frac{1}{3}$ and cutoff frequency $\omega_c = \frac{\pi}{4}$.
Hence, the fourier tranform of $x[n]$ is:

$$X(e^{j\omega}) = \begin{cases} \frac{1}{3}, & |\omega| \leq \frac{\pi}{4} \\ 0, & \text{else} \end{cases}$$

So:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{3}\right)^2 d\omega$$

$$= \frac{1}{18\pi} \left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{1}{36}$$

Exercise 2

A phase shifter at $90°$ can be seen as an LTI system with frequency Response:

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi \\ j, & -\pi < \omega < 0 \end{cases}$$

(a) Prove that

$$h[n] = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

(b) Show thay the above expression can be refactored to:

$$h[n] = \begin{cases} \frac{2}{\pi}\frac{\sin^2(\pi \frac{n}{2})}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

Solution

(a) Using the iDTFT formula:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) \cdot e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{0} j \cdot e^{j\omega n} d\omega$$

$$- \frac{1}{2\pi}\int_{0}^{\pi} j e^{j\omega n} d\omega = \frac{1}{2\pi n} \cdot e^{j\omega n}\Big|_{-\pi}^{0} - \frac{1}{2\pi n} \cdot e^{j\omega n}\Big|_{0}^{\pi}$$

$$= \frac{1}{2\pi n}(1 - e^{-j\pi n}) - \frac{1}{2\pi n}(e^{j\pi n} - 1) \qquad (1)$$

$$= \frac{1}{2\pi n}(1 - (-1)^n) + \frac{1}{2\pi n}(1 - (-1)^n)$$

$$= \frac{1}{\pi n} \cdot (1 - (-1)^n) \iff h[n] = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

(b)

Using (1):

$$h[n] = \frac{1}{2\pi n}\cdot\left(1-e^{-j\pi n}\right) - \frac{1}{2\pi n}\cdot\left(e^{j\pi n}-1\right)$$

$$= \underline{\frac{1}{2\pi n}} - \frac{1}{2\pi n}\cdot e^{-j\pi n} - \frac{1}{2\pi n}\cdot e^{j\pi n} + \underline{\frac{1}{2\pi n}}$$

$$= \frac{1}{\pi n} - \frac{1}{2\pi n}\left(e^{j\pi n}+e^{-j\pi n}\right)$$

$$= \frac{1}{\pi n} - \frac{1}{\pi n}\cdot\cos(\pi n) \qquad \text{(Euler)}$$

$$= \frac{1}{\pi n}\left(1-\cos(\pi n)\right)$$

$$= \frac{1}{\pi n}\, 2\sin^2\left(\frac{\pi}{2} n\right) \qquad \text{(Trigonometric Property)}$$

$$\Leftrightarrow h[n] = \begin{cases} \frac{2}{\pi}\cdot\frac{\sin^2\left(n\frac{\pi}{2}\right)}{n}, & n\neq 0 \\ 0, & n=0 \end{cases}$$