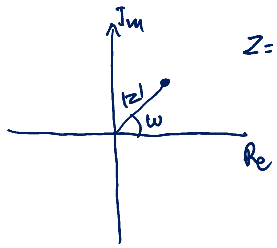
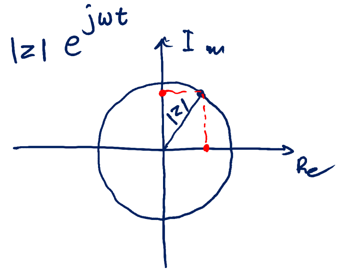
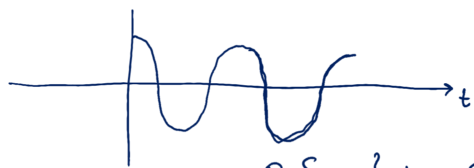
HY 215

$z = |z| \cdot e^{j\omega}$



$z(t) = |z|\, e^{j\omega t}$





$$x(t) = Re\{z(t)\} = |z| \cdot Re\{e^{j\omega t}\} = |z|\, Re\{\cos(\omega t) + j\sin(\omega t)\} =$$
$$= |z| \cos(\omega t)$$

$$x(t) = A\cos(\omega_0 t + \varphi) \;\Big|\; \Rightarrow x(t) = \underbrace{\overset{\text{Phasor}}{\frac{A}{2} e^{j\varphi}}}_{\text{Βάρη } X_{\omega_0}} \cdot \underbrace{e^{j\omega_0 t}}_{\text{συνάρτηση βάσης } \Psi_{\omega_0}} + \underbrace{\frac{A}{2} e^{-j\varphi}}_{X^*_{\omega_0}} \cdot \underbrace{e^{-j\omega_0 t}}_{\Psi^*_{\omega_0}} = X_{\omega_0} \cdot \Psi_{\omega_0} + X^*_{\omega_0} \Psi^*_{\omega_0}$$

Euler

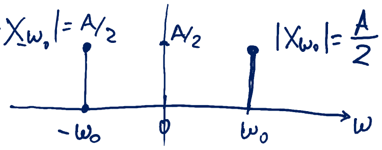
$$\cos\theta = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta}$$

X_{ω_0}: $|X_{\omega_0}|$, $\angle X_{\omega_0}$

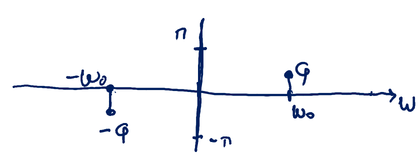
$$= \begin{bmatrix} X_{\omega_0} & X^*_{\omega_0} \end{bmatrix} \begin{bmatrix} \Psi_{\omega_0} \\ \Psi^*_{\omega_0} \end{bmatrix}$$

Αναπαράσταση Συχνότητας:

A) Μέτρου $|X_{\omega_0}|$



B) Φάσης



Παράδειγμα:

$$x(t) = 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right) = \frac{3}{2} e^{jt} + \frac{3}{2} e^{-jt} + \frac{1}{2j} e^{-j\pi/6} e^{j5t} - \frac{1}{2j} e^{+j\pi/6} e^{-j5t} - \left(e^{-j\pi/3} e^{j8t} + e^{j\pi/3} \cdot e^{-j8t}\right)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\frac{1}{2j} = \frac{j}{2j^2} = -\frac{j}{2} = \frac{1}{2}(-j) = \frac{1}{2} e^{-j\pi/2}$$

$$\frac{1}{2j} e^{-j\pi/6} = \frac{1}{2} e^{-j\pi/2} \cdot e^{-j\pi/6} = \frac{1}{2} e^{-j\left(\frac{3\pi}{6} + \frac{\pi}{6}\right)} = \frac{1}{2} e^{-j4\pi/6} = \frac{1}{2} e^{-j2\pi/3} \quad (1)$$

$$-e^{-j\pi/3} = (-1) e^{-j\pi/3} = e^{j\pi} e^{-j\pi/3} = e^{j(\pi - \pi/3)} = e^{j2\pi/3} \quad (2)$$

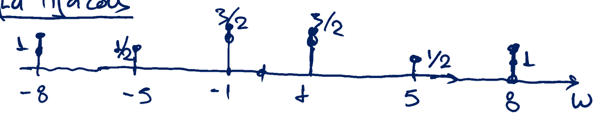
$$-e^{j\pi/3} = (-1) e^{j\pi/3} = e^{j\pi} \cdot e^{j\pi/3} = e^{j4\pi/3} = e^{-j2\pi/3} \quad (3) \qquad \frac{4\pi}{3} \bmod_{2\pi} = \frac{4\pi}{3} - 2\pi = \frac{4\pi}{3} - \frac{6\pi}{3} = -\frac{2\pi}{3}$$

Άρα: $x(t) = \boxed{e^{-j2\pi/3}} e^{-j8t} + \boxed{\tfrac{1}{2} e^{j2\pi/3}} e^{-j5t} + \boxed{\tfrac{3}{2}} e^{-jt} + \boxed{\tfrac{3}{2}} e^{jt} + \boxed{\tfrac{1}{2} e^{-j2\pi/3}} e^{j5t} + \boxed{e^{j2\pi/3}} e^{j8t}$

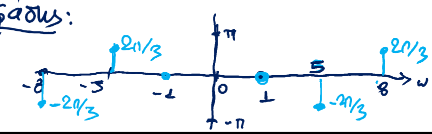
βάρη

συναρτήσεις βάσης

Φάσμα πλάτους



Φάσμα φάσης:



Γενικότερα:

$$x(t) = \sum_{k=0}^{L} A_k \cos(\omega_k t + \varphi_k) = \sum_{k=-L}^{L} \frac{A_k}{2} \cdot e^{j\varphi_k} \cdot e^{j\omega_k t} = (1)$$

$$\boxed{A_k = A_{-k} \quad \omega_k = -\omega_{-k} \quad \varphi_k = -\varphi_{-k}}$$

Euler: $\cos\theta = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta}$

παράδειγμα $k = 3$:

$$A_3 \cos(\omega_3 t + \varphi_3) = \underbrace{\frac{A_3}{2} \cdot e^{-j\varphi_3} \cdot e^{-j\omega_3 t}}_{\frac{A_{-3}}{2} e^{j\varphi_{-3}} \cdot e^{j\omega_{-3} t}} + \frac{A_3}{2} e^{j\varphi_3} \cdot e^{j\omega_3 t}$$

$$= \frac{A_{-3}}{2} e^{j\varphi_{-3}} \cdot e^{j\omega_{-3} t} + \frac{A_3}{2} e^{j\varphi_3} \cdot e^{j\omega_3 t}$$

(1) $$x(t) = \underbrace{\frac{A_{-L}}{2} e^{j\varphi_{-L}}}_{X_{-L}} \underbrace{e^{j\omega_{-L} t}}_{\psi_{-L}(t)} + \dots + \underbrace{\frac{A_{-1}}{2} e^{j\varphi_{-1}}}_{X_{-1}} \underbrace{e^{j\omega_{-1} t}}_{\psi_{-1}(t)} + \underbrace{\frac{A_0}{2} e^{j\varphi_0}}_{X_0} \cdot \underbrace{e^{j\omega_0 t}}_{\psi_0(t)} + \underbrace{\frac{A_1}{2} e^{j\varphi_1}}_{X_1} \cdot \underbrace{e^{j\omega_1 t}}_{\psi_1(t)} + \dots$$

$$\dots \underbrace{\frac{A_L}{2} e^{j\varphi_L}}_{X_L} \cdot \underbrace{e^{j\omega_L t}}_{\psi_L(t)} \Rightarrow$$

$\varphi_0 = 0$, $\omega_0 = 0$ | $\psi_0(t) = 1$, $e^{j\varphi_0} = 1$

$$\Rightarrow x(t) = X_{-L}\, \psi_{-L}(t) + \dots + X_{-1}\, \psi_{-1}(t) + X_0\, \cancel{\psi_0(t)} + X_1\, \psi_1(t) + \dots + X_L\, \psi_L(t)$$

Ορισμός εσωτερικού γινομένου:

$$\psi_k(t) = e^{j\omega_k t} \qquad \omega_k = k\omega_0 \quad \Big| \Rightarrow \psi_k(t) = e^{jk\omega_0 t}$$

$$\psi_k(t) = \psi_k(t+T_0) \quad \text{όπου} \quad T_0 = \frac{2\pi}{\omega_0}$$

$$\langle \psi_k(t), \psi_\ell(t) \rangle_{(t_1,t_2)} = \int_{t_1}^{t_2} \psi_k(t)\cdot\psi_\ell^*(t)\,dt$$

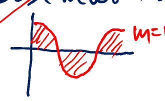
όταν $\psi_k(t) = e^{jk\omega_0 t}$

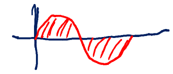
$k,\ell \in \mathbb{Z}$

$k-\ell = m \in \mathbb{Z}$

$$\langle \psi_k(t), \psi_\ell(t) \rangle_{(0,T_0)} = \int_0^{T_0} \psi_k(t)\cdot\psi_\ell^*(t)\,dt = \int_0^{T_0} e^{jk\omega_0 t}\cdot e^{-j\ell\omega_0 t}\,dt = \int_0^{T_0} e^{j(k-\ell)\omega_0 t}\,dt =$$

$$k\neq\ell \quad \int_0^{T_0} e^{jm\omega_0 t}\,dt = \int_0^{T_0} \cancelto{0}{\cos(m\omega_0 t)\,dt} + j\int_0^{T_0} \cancelto{0}{\sin(m\omega_0 t)\,dt} = 0$$





$$k=\ell \quad \int_0^{T_0} e^{j0t}\,dt = \int_0^{T_0} dt = T_0$$

$$\Rightarrow \boxed{\int_0^{T_0} \psi_k(t)\cdot\psi_\ell^*(t)\,dt = \begin{cases} 0, & k\neq\ell \\ T_0, & k=\ell \end{cases}}$$

$\langle \psi_k(t), \psi_\ell(t) \rangle$

$$x(t) = X_{-L}\,\psi_{-L}(t) + X_{-(L-1)}\,\psi_{-(L-1)}(t) + \cdots + X_{-1}\psi_{-1}(t) + X_0 + X_1\psi_1(t) + \cdots X_{L-1}\psi_{L-1}(t) + X_L\cdot\psi_L(t)$$

$x(t) = x(t+T_0)$

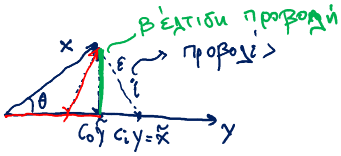
$\psi_k(t) = e^{jk\omega_0 t}$

$$\langle x(t), \psi_1(t) \rangle_{(0,T_0)} = \int_0^{T_0} x(t)\cdot\psi_1^*(t)\,dt = \int_0^{T_0} x(t)\cdot e^{-j\omega_0 t}\,dt$$

$$\langle \sum_{k=-L}^{L} X_k\,\psi_k(t), \psi_1(t) \rangle = \sum_{k=-L}^{L} X_k \underbrace{\langle \psi_k(t), \psi_1(t) \rangle}_{\begin{cases} 0 & k\neq 1 \\ T_0 & k=1 \end{cases}} = X_1\cdot T_0 \Rightarrow$$

$$\Rightarrow X_1 = \frac{1}{T_0}\langle x(t), \psi_1(t) \rangle_{(0,T_0)}$$ και γενικότερα:

$$\boxed{X_k = \frac{1}{T_0}\langle x(t), \psi_k(t) \rangle_{(0,T_0)}}$$



$$\cos\theta = \frac{c_0 |y|}{|x|} \Rightarrow c_0 = \frac{|x|}{|y|}\cos\theta = \frac{1}{|y|^2}\underbrace{\left(|x|\cdot|y|\cdot\cos\theta\right)}_{\langle \vec{x},\vec{y}\rangle} \Rightarrow$$

$\langle \vec{x},\vec{y}\rangle$ εσωτερικό γινόμενο.

$$\varepsilon = x - \tilde{x} \Rightarrow x = \tilde{x} + \varepsilon \qquad \Rightarrow c_0 = \frac{1}{|y|^2}\langle \vec{x},\vec{y}\rangle$$

Από τα διανύσματα σε σήματα:

$x(t) \underset{\text{προβ.}}{\rightsquigarrow} y(t)$ λάθος $\epsilon(t) = x(t) - c_0\cdot y(t)$ προσέγγιση του $x(t)$ ως προς $y(t)$ στο διάστημα (t_1, t_2)

Συνολικό σφάλμα: $E = \int_{t_1}^{t_2} \epsilon^2(t)\,dt = \int_{t_1}^{t_2}\left(x(t) - c_0\, y(t)\right)^2 dt$

$\frac{d}{dc}E = 0$ αυτό δίνει τη βέλτιστη τιμή του $c_0 = \hat{c}_0$ (βέλτιστο βάρος)

Για πραγματικά σήματα

Λύση: $\hat{c}_0 = \frac{1}{\|y(t)\|^2}\int_{t_1}^{t_2} x(t)\cdot y(t)\,dt \Rightarrow \hat{c}_0 = \frac{1}{\|y(t)\|^2}\langle x(t), y(t)\rangle$

Συνοπτικά: (Α) Συναρτήσεις βάσης $\psi_k(t) = e^{jk\omega_0 t}$, $\psi_k(t) = \psi_k(t+T_0)$

$T_0 = \frac{2\pi}{\omega_0}$

(Β) $\langle \psi_k(t), \psi_\ell(t)\rangle = \int_0^{T_0}\psi_k(t)\cdot\psi_\ell^*(t)\,dt = \begin{cases} 0 & k\neq \ell \\ T_0 & k=\ell \end{cases}$

(Γ) $x(t) = \sum_{k=-L}^{L} \underbrace{A_k\cdot e^{j\varphi_k}}_{X_k}\cdot\underbrace{e^{jk\omega_0 t}}_{\psi_k(t)} = \sum_{k=-L}^{L} X_k\cdot\psi_k(t)$

$\hookrightarrow \{\psi_k(t)\}$ οικογ. συναρ. βάσης $\rightsquigarrow$ χώρος περιοδ. συναρτήσεων με περίοδο T_0

$x(t+T_0) = x(t)$

Προβολή του $x(t)$ σε αυτό το χώρο: (έστω στην κατεύθυνση της $\psi_\ell(t)$)

$$\langle x(t), \psi_\ell(t)\rangle_{(0,T_0)} = \int_0^{T_0} x(t)\cdot\psi_\ell^*(t)\,dt = \int_0^{T_0}\sum_{k=-L}^{L} X_k\,\psi_k(t)\cdot\psi_\ell^*(t)\,dt =$$

$$= \sum_{k=-L}^{L} X_k \underbrace{\int_0^{T_0}\psi_k(t)\cdot\psi_\ell^*(t)\,dt}_{T_0\ \ k=\ell,} = X_\ell\cdot T_0 \Rightarrow X_\ell = \frac{1}{T_0}\langle x(t), \psi_\ell(t)\rangle_{(0,T_0)}$$ ή γενικότερα:

$$\boxed{X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jk\omega_0 t}\,dt}$$

Άρα:

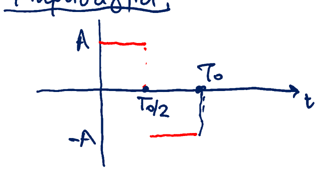
Αν $\boxed{x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot e^{jk\omega_0 t}}$ Ανάπτυγμα σε σειρά Fourier , $x(t)=x(t+T_0)$ περιοδικό.

τότε $$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jk\omega_0 t}\,dt \leftarrow$$

$$X_0 = \frac{1}{T_0}\int_0^{T} x(t)\,dt$$

$\boxed{\text{Ανάπτυγμα περιοδικού σήματος σε σειρά Fourier}}$

Παράδειγμα:



$$x(t) = \begin{cases} A, & 0 \le t < T_0/2 \\ -A, & \frac{T_0}{2} \le t < T_0 \end{cases}$$

$$x(t) = x(t+T_0) \Rightarrow x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot e^{jk\omega_0 t}\,dt$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\left[\int_0^{T_0/2} A\,dt + \int_{T_0/2}^{T_0} (-A)\,dt\right] = \underbrace{\frac{A}{T_0}\int_0^{T_0/2} dt}_{A/2} - \underbrace{\frac{A}{T_0}\int_{T_0/2}^{T_0} dt}_{A/2} = 0$$

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jk\omega_0 t}\,dt = \frac{A}{T_0}\int_0^{T_0/2} e^{-jk\frac{2\pi}{T_0}t}\,dt - \frac{A}{T_0}\int_{T_0/2}^{T_0} e^{-jk\frac{2\pi}{T_0}t}\,dt =$$

$$= \frac{A}{\cancel{T_0}}\,\frac{\cancel{T_0}}{-jk2\pi}\, e^{-jk\frac{2\pi}{T_0}t}\Big|_0^{T_0/2} - \frac{A}{\cancel{T_0}}\,\frac{\cancel{T_0}}{-jk2\pi}\, e^{-jk\frac{2\pi}{T_0}t}\Big|_{T_0/2}^{T_0} = \frac{A}{-jk2\pi}\left(e^{-jk\frac{2\pi}{\cancel{T_0}}\frac{\cancel{T_0}}{2}} - 1\right) +$$

$$+ \frac{A}{jk2\pi}\left(e^{-jk\frac{2\pi}{\cancel{T_0}}\cancel{T_0}} - e^{-jk\frac{2\pi}{\cancel{T_0}}\frac{\cancel{T_0}}{2}}\right) = \frac{jA}{k\cdot 2\pi}\left(e^{-jk\pi} - 1\right) - \frac{jA}{k\cdot 2\pi}\left(e^{-jk2\pi} - e^{-jk\pi}\right) =$$

$$= \frac{jA}{k2\pi}\left(\underbrace{e^{-jk\pi}}_{(e^{-j\pi})^k} - 1 \underbrace{- e^{-jk2\pi}}_{-1} + e^{-jk\pi}\right) = \frac{jA}{k2\pi}\left(2\left(e^{-j\pi}\right)^k - 2\right) = \frac{jA}{k\pi}\left(\left(e^{-j\pi}\right)^k - 1\right) = \frac{jA}{k\pi}\left((-1)^k - 1\right)$$

$$= \begin{cases} 0 & k \text{ άρτιος} \\ -j\frac{2A}{k\pi} & k \text{ περιττά.} \end{cases}$$

Ιδιότητες

$$x(t) = \sum_k X_k \cdot e^{jk\omega_0 t} \qquad (x(t) = x(t+T_0))$$

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jk\omega_0 t}\,dt$$

• $x(t-t_0)$

$$\tilde{X}_k = \frac{1}{T_0}\int_0^{T_0} x(t-t_0)\cdot e^{-jk\omega_0 t}\,dt = \underbrace{\frac{1}{T_0}\int_0^{T_0} x(\tau)\, e^{-jk\omega_0 \tau}\,d\tau}_{X_k} \cdot e^{-jk\omega_0 t_0} \Rightarrow$$

$t - t_0 = \tau$

$t = \tau + t_0$

$$\Rightarrow \tilde{X}_k = X_k \cdot e^{-jk\omega_0 t_0}$$

• $x(-t)$

$$\tilde{X}_k = \frac{1}{T_0}\int_0^{T_0} x(-t)\, e^{-jk\omega_0 t}\,dt = \underbrace{\frac{1}{T_0}\int^{T_0} x(\tau)\cdot e^{jk\omega_0 t}\,dt}_{X_{-k}} = X_{-k}$$

$\tau = -t$

$$x(t) = \sum_k X_k\, e^{jk\omega_0 t} \qquad x(t) = x(t+T_0)$$

$$y(t) = \sum_k Y_k \cdot e^{jk\omega_0 t} \qquad y(t) = y(t+T_0)$$

$$\Delta \quad \int_0^{T_0} x(t)\cdot y^*(t)\,dt = \int_0^{T_0} \sum_k X_k \cdot e^{jk\omega_0 t} \cdot \sum_\ell Y_\ell^* e^{-j\ell\omega_0 t}\,dt = \sum_{k,\ell} X_k Y_\ell^* \cdot \underbrace{\int_0^{T_0} e^{j(k-\ell)\omega_0 t}\,dt}_{\substack{k=\ell \\ T_0}}$$

$$\Rightarrow \int_0^{T_0} x(t)\, y^*(t)\,dt = \sum_k X_k \cdot Y_k^* \cdot T_0 \Rightarrow \boxed{\frac{1}{T_0}\int_0^{T_0} x(t)\, y^*(t)\,dt = \sum_k X_k Y_k^*}$$

Άρα αν $y(t) = x(t)$

$$\frac{1}{T_0}\int_0^{T_0} x(t)\, x^*(t)\,dt = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt = \sum_k X_k \cdot X_k^* = \sum_k |X_k|^2$$

Δηλ. $\boxed{\frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt = \sum_k |X_k|^2}$ Θεώρημα Parseval