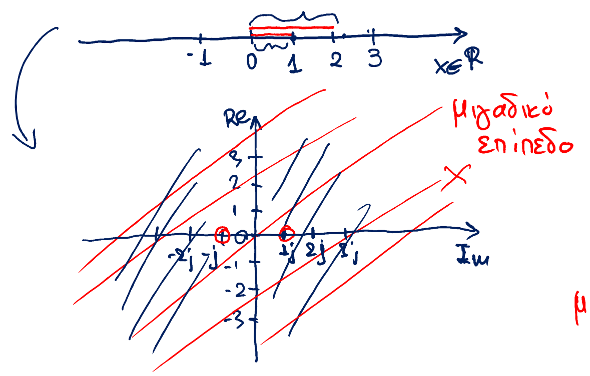
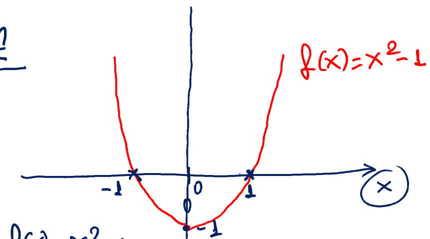
HY 215

Διάλεξη 2η

$|x| \rightarrow |x-0| = |x|$





$f(x) = x^2 - 1$

$f(x) = 0 \Rightarrow$

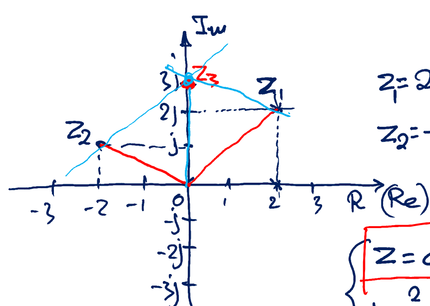
$x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \quad (\pm\sqrt{1})$

$f(x) = x^2 + 1$

μιγαδικοί $f(x) = 0 \Rightarrow x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\underbrace{\sqrt{-1}}_{j} \Rightarrow$

$\Rightarrow x = \pm j$ όπου $\boxed{j^2 = -1}$

φανταστικός

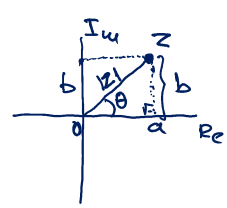


$z_1 = 2 + j2$

$z_2 = -2 + j$

$z_3 = z_1 + z_2 = (2 + j2) + (-2 + j) = (2 - 2) + j(2 + 1) =$

$= 0 + 3j$



$\boxed{z = a + jb}$ Καρτεσιανή μορφή

$|z|^2 = a^2 + b^2 \Rightarrow |z| = \sqrt{a^2 + b^2}$

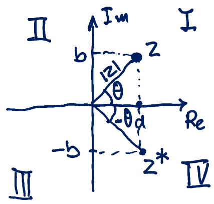
$\cos\theta = \dfrac{a}{|z|}, \quad \sin\theta = \dfrac{b}{|z|}$

$\theta = \tan^{-1}\left(\dfrac{b}{a}\right) \quad \theta \in [0, 2\pi)$

$\rightarrow z = |z|\cdot\cos\theta + j\,|z|\cdot\sin\theta \Rightarrow z = |z|\underbrace{(\cos\theta + j\sin\theta)}_{\text{Euler}}$

$\Rightarrow \boxed{z = |z|\cdot e^{j\theta}}$ πολική μορφή

$e^{j\theta} = \cos\theta + j\sin\theta$



$z = a + jb \Rightarrow z = |z| \cdot e^{j\theta}$

$|z| = \sqrt{a^2+b^2}$ (μέτρο)

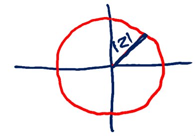
$\theta = \tan^{-1}\frac{b}{a}$ (φάση)

$= \angle z$

$\boxed{\text{Γεωμ. τόπος}}$

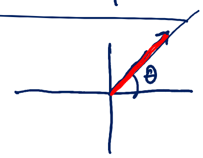
$|z|$ σταθερό

θ μεταβλητό



$|z|$ μεταβλητό

θ σταθερό



$\boxed{\text{συζυγής του } z:}$

$z^* = a - jb$

π.χ. $z = 3 - j2 \Rightarrow z^* = 3 + j2$

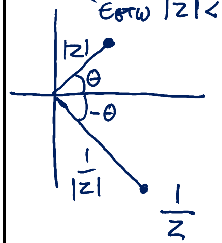
$|z^*| = \sqrt{a^2+b^2} = |z|$

$\angle z^* = \tan^{-1}\frac{-b}{a} = -\tan^{-1}\frac{b}{a} = -\angle z$

$z^* = |z| \cdot e^{-j\theta}$

Αντίστροφο

Έστω $|z|<1$



Αντίστροφος

$z_1 = \frac{1}{z} = \frac{1}{a+jb}$

$= \frac{1}{|z| \cdot e^{j\theta}} = \frac{1}{|z|} e^{-j\theta}$

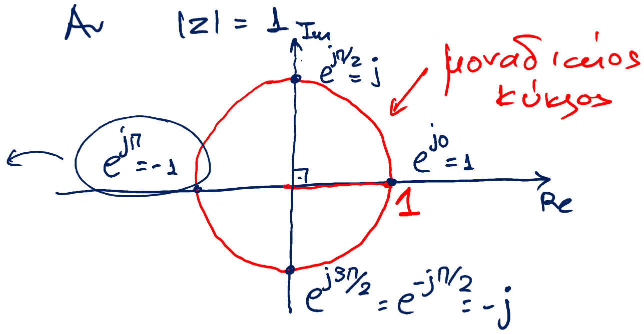
Γεωμ. τόπος (Γ.Τ.) (σημαντική ειδική περίπτωση)

$z = a + jb$

όταν $|z|$ σταθερό, θ μεταβλητό, τότε ο Γ.Τ. είναι ένας κύκλος ακτίνας $|z|$

Αν $|z| = 1$

$\boxed{e^{j\pi} + 1 = 0}$



Euler $e^{j\theta} = \cos\theta + j\sin\theta$

$$5e^{j\theta} = 5\cos\theta + j5\sin\theta \quad \Bigg\} \quad 5\overset{1}{\cancel{\cos 0}} + j5\overset{0}{\cancel{\sin 0}} = 5 \quad \text{δηλ.} \quad 5 = 5\cdot e^{j0}$$

$: \theta = 0$

$\theta=\pi$: $$5e^{j\pi} = 5\overset{-1}{\cancel{\cos\pi}} + j5\overset{0}{\cancel{\sin\pi}} = -5 = 5\cdot e^{j\pi}$$

$$e^{j\pi/2} = j$$

$$j\cdot j = e^{j\pi/2}\cdot e^{j\pi/2} = e^{j\pi} = -1 \Rightarrow \boxed{j^2 = -1}$$

Πράξεις με μιγαδικούς

$z_1 = a_1 + jb_1$, $z_2 = a_2 + jb_2$

$z_1 = |z_1|\cdot e^{j\theta_1}$, $z_2 = |z_2|\cdot e^{j\theta_2}$

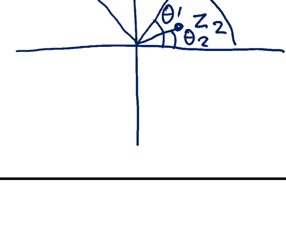
▷ Πρόσθεση:

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \quad \Big| \quad |z|\cdot e^{j\theta}$$
$$= |z_1|\cdot e^{j\theta_1} + |z_2|\cdot e^{j\theta_2}$$

▷ Πολλαπλασιασμός:

$$z_1\cdot z_2 = (a_1 + jb_1)\cdot(a_2 + jb_2) = (a_1a_2 - b_1b_2) + j(b_1a_2 + a_1b_2)$$

$$z_1\cdot z_2 = |z_1|\cdot e^{j\theta_1}\cdot|z_2|\cdot e^{j\theta_2} = \underbrace{|z_1|\cdot|z_2|}_{|z|}\cdot e^{j\underbrace{(\theta_1+\theta_2)}_{\theta}} = |z|\cdot e^{j\theta}$$



$\theta_1 > 45$
$\theta_2 > 45$
$|z_1|, |z_2| > 1$

Πράξεις ....

▸ Πολλαπλασιασμός

$$z_1 = a_1 + j b_1$$

$$z_1 \cdot z_1^* = (a_1 + j b_1)(a_1 - j b_1) = a_1^2 + b_1^2 = |z_1|^2$$

$$z_1 \cdot z_1^* = |z_1| \cdot e^{j\theta} \cdot |z_1| \cdot e^{-j\theta} = |z_1|^2$$

▸ Διαίρεση:

$$\frac{z_1}{z_2} = \frac{a_1 + j b_1}{a_2 + j b_2} = \frac{(a_1 + j b_1)(a_2 - j b_2)}{(a_2 + j b_2)(a_2 - j b_2)} = \frac{(a_1 a_2 + b_1 b_2) + j(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}.$$

$$= \frac{|z_1| \cdot e^{j\theta_1}}{|z_2|\, e^{j\theta_2}} = \frac{|z_1|}{|z_2|} \cdot e^{j(\theta_1 - \theta_2)} \quad \checkmark$$

▸ Πρόσθεση συζυγ.

$z = a + jb$ $\qquad Re\{z\} = a$

$z^* = a - jb$ $\qquad Im\{z\} = b$

Euler:

$$\boxed{e^{j\theta} = \cos\theta + j \sin\theta}$$

$$z + z^* = 2a = 2\, Re\{z\}$$

$$\left.\begin{aligned} z &= e^{j\theta} \\ z^* &= e^{-j\theta} \end{aligned}\right\} \; z + z^* = e^{j\theta} + e^{-j\theta} = \overbrace{(\cos\theta + j\sin\theta)}^{e^{j\theta}} + \overbrace{(\cos(-\theta) + j\sin(-\theta))}^{e^{-j\theta}}$$

$$= \cos\theta + \cancel{j\sin\theta} + \cos\theta - \cancel{j\sin\theta} = 2\cos\theta \Rightarrow$$

$$\Rightarrow \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$z - z^* = e^{j\theta} - e^{-j\theta} = 2j\sin\theta \Rightarrow$$

$$\Rightarrow \boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

Αντίστροφες σχέσεις του Euler

▹ Δυνάμεις $\quad z = a + jb$
$\qquad\qquad\;\; = |z|\cdot e^{j\theta}$

$$z^N = \left(|z|\cdot e^{j\theta}\right)^N = |z|^N\cdot e^{jN\theta} = |z|^N\cdot\Big(\underbrace{\cos(N\theta) + j\sin(N\theta)}_{e^{jN\theta} = \cos N\theta + j\sin N\theta}\Big)$$

De-Moivre

▹ Modulo.

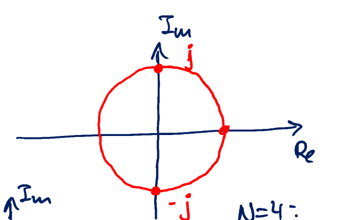
$$z_1 = |z_1|\cdot e^{j\theta_1}$$
$$z_2 = |z_2|\cdot e^{j\theta_2} = |z_1|\cdot e^{j(\theta_1 + k2\pi)} = |z_1|\cdot e^{j\theta_1}\cdot \cancel{e^{jk2\pi}}^{\,1} = |z_1|\cdot e^{j\theta_1} = z_1$$

όπου $\theta_2 = \theta_1 + k\cdot 2\pi$, $\quad k\in\mathbb{Z}$

$$\theta_1 +_{2\pi} \theta_2 = \theta \in [0, 2\pi)$$

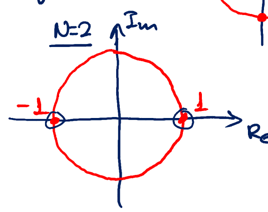
$22 + 3 = 25$
$22 +_{24} 3 = 1$

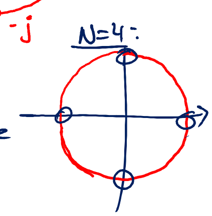
Εξισώσεις μιγαδικών.



$$\left.\begin{aligned} f(z) &= z^2 + 1 \\ f(z) &= 0 \end{aligned}\right\} z^2 + 1 = 0 \Rightarrow z^2 = -1 \Rightarrow \begin{cases} z_1 = j \\ z_2 = -j \end{cases}$$

$$\left.\begin{aligned} f(z) &= z^2 - 1 \\ f(z) &= 0 \end{aligned}\right\} \Rightarrow z^2 = 1 \Rightarrow \begin{cases} z_1 = 1 \\ z_2 = -1 \end{cases}$$





Νιοστές ρίζες της μονάδας:

$$\left.\begin{aligned} f(z) &= z^N - 1 \\ f(z) &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &z^N - 1 = 0 \Rightarrow z^N = 1 \\ &z = |z|\cdot e^{j\theta} \\ &1 = e^{j0} = e^{j2k\pi} \end{aligned}\right\} \left(|z|\cdot e^{j\theta}\right)^N = e^{j2k\pi} \Rightarrow$$

$$\Rightarrow |z|^N\cdot e^{jN\theta} = e^{j2k\pi} \Rightarrow \begin{cases} |z|^N = 1 \Rightarrow |z| = 1 \\ N\theta = 2k\pi \Rightarrow \theta_k = \frac{2\pi}{N}k \end{cases}$$

Λύσεις $\Rightarrow z_k = e^{j\frac{2\pi}{N}k}$, $\quad k = 0, 1, 2, \ldots, N-1$

$N=2$, $\quad k = 0, 1$
$N=4$, $\quad k = 0, 1, 2, 3$
$\frac{2\pi}{4} = \frac{\pi}{2}$