ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ
Τμήμα Επιστήμης Υπολογιστών

ΗΥ-215: Εφαρμοσμένα Μαθηματικά για Μηχανικούς
Εαρινό Εξάμηνο 2016-17
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Άσκηση 1

(i) $\frac{1}{2}e^{j\pi} = \frac{1}{2}\cos(\pi) + \frac{1}{2}j\sin(\pi) = -\frac{1}{2}$

(ii) $\frac{1}{2}e^{-j\pi} = \frac{1}{2}\cos(-\pi) + \frac{1}{2}j\sin(-\pi) = -\frac{1}{2}$,ίδιο με (i)

(iii) $e^{j\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + j\sin(\frac{\pi}{2}) = j$

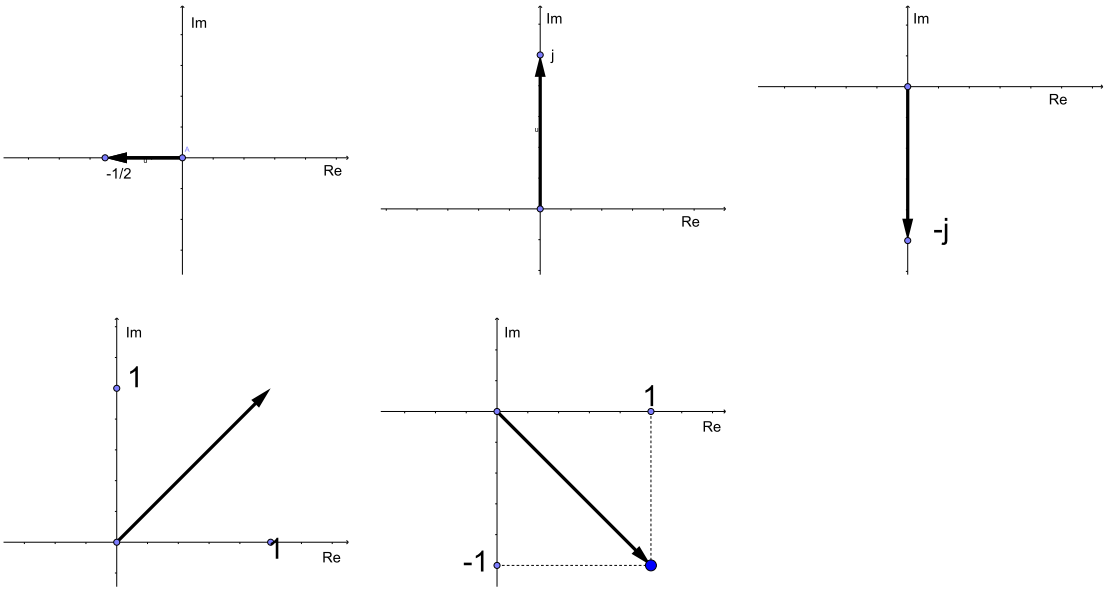
(iv) $e^{-j\frac{\pi}{2}} = \cos(\frac{-\pi}{2}) + j\sin(-\frac{\pi}{2}) = -j$

(v) $e^{j\frac{5\pi}{2}} = \cos(\frac{5\pi}{2}) + j\sin(\frac{5\pi}{2}) = \cos(\frac{4\pi+\pi}{2}) + j\sin(\frac{4\pi+\pi}{2}) = \cos(\frac{\pi}{2}) + j\sin(\frac{\pi}{2}) = j$, ίδιο με (iii)

(vi) $\sqrt{2}e^{j\frac{\pi}{4}} = \sqrt{2}\cos(\frac{\pi}{4}) + \sqrt{2}j\sin(\frac{\pi}{4}) = \sqrt{2}\frac{\sqrt{2}}{2} + \sqrt{2}j\frac{\sqrt{2}}{2} = 1 + j$

(vii) $\sqrt{2}e^{j\frac{9\pi}{4}} = \sqrt{2}e^{j\frac{8\pi+\pi}{4}} = \sqrt{2}e^{j\frac{\pi}{4}} = 1 + j$ ίδιο με (vi)

(viii) $\sqrt{2}e^{-j\frac{9\pi}{4}} = \sqrt{2}e^{-j\frac{8\pi+\pi}{4}} = \sqrt{2}e^{-j\frac{\pi}{4}} = \sqrt{2}\cos(\frac{-\pi}{4}) + \sqrt{2}j\sin(\frac{-\pi}{4}) = 1 - j$



Σχήμα 1: 1i,iii,iv,vi,viii

Άσκηση 2

(i) $5 = 5e^{j0}$

(ii) $-2 = 2e^{j\pi}$

(iii) $-3j = 3e^{-j\frac{\pi}{2}}$

(iv) $\frac{1}{2} - j\frac{\sqrt{3}}{2}, \qquad |z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1, \quad \theta = \tan^{-1}(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}, \text{ άρα } e^{-j\frac{\pi}{3}}$

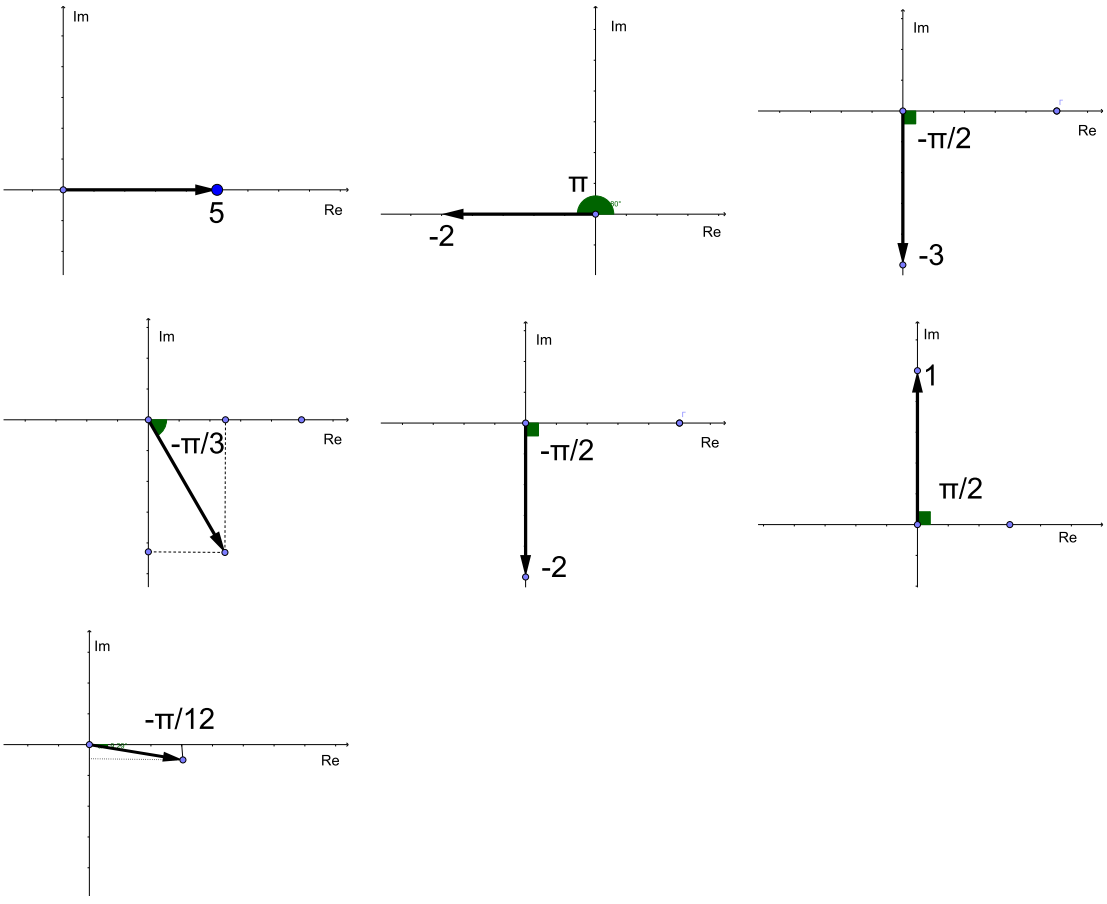
(v) $1 + j, \qquad |z| = \sqrt{1+1} = \sqrt{2}, \qquad \theta = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}, \text{ άρα } \sqrt{2}e^{j\frac{\pi}{4}}$

(vi) $(1-j)^2 = 1 - 2j + j^2 = 1 - 2j - 1 = -2j = 2e^{-j\frac{\pi}{2}}$

(vii) $j(1-j) = j - j^2 = j + 1 = 1 + j = \sqrt{2}e^{j\frac{\pi}{4}}$, ίδιο με (v)

(viii) $\frac{1+j}{1-j} = \frac{\sqrt{2}e^{j\frac{\pi}{4}}}{\sqrt{2}e^{-j\frac{\pi}{4}}} = e^{j\frac{\pi}{4}}e^{j\frac{\pi}{4}} = e^{j\frac{\pi}{2}}$

(ix) $\frac{\sqrt{2}+j\sqrt{2}}{1+j\sqrt{3}} = \frac{\sqrt{2}(1+j)}{2(\frac{1}{2}+j\frac{\sqrt{3}}{2})} = \frac{\sqrt{2}}{2}\frac{\sqrt{2}e^{j\frac{\pi}{4}}}{e^{j\frac{\pi}{3}}} = e^{j\frac{\pi}{4}}e^{-j\frac{\pi}{3}} = e^{-j\frac{\pi}{12}}$



Σχήμα 2: 2i,ii,iii,iv,vi,viii,ix

Άσκηση 3

$$\Re\left\{(1+j)e^{j\theta}\right\} = -1$$
$$\Re\left\{\sqrt{2}e^{j\frac{\pi}{4}}e^{j\theta}\right\} = -1$$
$$\Re\left\{\sqrt{2}e^{j(\theta+\frac{\pi}{4})}\right\} = -1$$
$$\Re\left\{e^{j(\theta+\frac{\pi}{4})}\right\} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} = \cos(\frac{3\pi}{4})$$
$$\cos(\theta+\frac{\pi}{4}) = \cos(\frac{3\pi}{4})$$

Άρα

$$\theta + \frac{\pi}{4} = 2k\pi \pm \frac{3\pi}{4} \Leftrightarrow \begin{Bmatrix}\theta = 2k\pi + \frac{\pi}{2}, k \in \mathrm{Z}\\ \theta = 2k\pi - \pi, k \in \mathrm{Z}\end{Bmatrix}$$

Άσκηση 4

(α') Είναι

$$5\cos(2\pi f_0 t) - 4\cos(2\pi f_0 t + \psi) = M\cos(2\pi f_0 t + \frac{\pi}{4})$$
$$\Re\left\{5e^{j2\pi f_0 t}\right\} - \Re\left\{4e^{j2\pi f_0 t}e^{j\psi}\right\} = \Re\left\{Me^{j2\pi f_0 t}e^{j\frac{\pi}{4}}\right\}$$
$$\Re\left\{(5-4e^{j\psi})e^{j2\pi f_0 t}\right\} = \Re\left\{Me^{j\frac{\pi}{4}}e^{j2\pi f_0 t}\right\}$$
$$5 - 4e^{j\psi} = Me^{j\frac{\pi}{4}}$$

Από την παραπάνω ισότητα,

$$\begin{cases} 5 - 4\cos\psi = M\cos\frac{\pi}{4} & = \frac{\sqrt{2}}{2}M \\ -4j\sin\psi = Mj\sin\frac{\pi}{4} & = j\frac{\sqrt{2}}{2}M \end{cases} \Rightarrow \begin{cases} 5 - 4\cos\psi & = \frac{\sqrt{2}}{2}M \\ -4\sin\psi & = \frac{\sqrt{2}}{2}M \end{cases} \Rightarrow \begin{cases} 5 & = 4\cos\psi + \frac{\sqrt{2}}{2}M \\ 0 & = 4\sin\psi + \frac{\sqrt{2}}{2}M \end{cases}$$

(β') Αφαιρούμε κατά μέλη, οπότε

$$5 - 0 = 4(\cos\psi - \sin\psi) \Leftrightarrow \frac{5}{4} = \cos\psi - \sin\psi \Leftrightarrow \frac{5}{4} = \Re\left\{(1+j)(\cos\psi + j\sin\psi)\right\} \Leftrightarrow \frac{5}{4} = \Re\left\{\sqrt{2}e^{j\frac{\pi}{4}}e^{j\psi}\right\}$$

και άρα

$$\frac{5}{4\sqrt{2}} = \cos(\psi + \frac{\pi}{4}) \Leftrightarrow \psi = \begin{Bmatrix}-0.405\pi \\ -0.095\pi\end{Bmatrix}$$

και

$$M = -4\sqrt{2}\sin\psi \Leftrightarrow M = \begin{Bmatrix}5.406 \\ 1.665\end{Bmatrix}$$

Άσκηση 5

Είναι

$$x(t) = 7\cos(2\pi f_0 t + \frac{\pi}{2}) + 6\cos(2\pi f_0 t) = 7\Re\left\{e^{j\frac{\pi}{2}}e^{j2\pi f_0 t}\right\} + 6\Re\left\{e^{j2\pi f_0 t}\right\} = \Re\left\{(7e^{j\frac{\pi}{2}} + 6)e^{j2\pi f_0 t}\right\}$$

όπου

$$z = 7e^{j\frac{\pi}{2}} + 6 = 6 + 7j$$

με

$$|z| = \sqrt{6^2 + 7^2} = \sqrt{36 + 49} \simeq 9.22$$

και

$$\theta = \tan^{-1} \frac{6}{7} = 0.862 \text{ rad}$$

Άρα

$$z = 9.22e^{j0.862}$$

οπότε

$$x(t) = \Re\left\{9.22e^{j0.862}e^{j2\pi f_0 t}\right\} = 9.22\cos(2\pi f_0 t + 0.8622)$$

Άσκηση 6

Είναι

$$x(t) = \sqrt{3}\cos(2\pi f_0 t + \frac{\pi}{3}) + \sin(2\pi f_0 t + \frac{\pi}{2}) = \sqrt{3}\cos(2\pi f_0 t + \frac{\pi}{3}) + \cos(2\pi f_0 t) = \Re\left\{(\sqrt{3}e^{j\frac{\pi}{3}} + 1)e^{j2\pi f_0 t}\right\}$$

όπου

$$z = 1 + \sqrt{3}e^{j\frac{\pi}{3}} = 1 + \sqrt{3}\cos(\frac{\pi}{3}) + \sqrt{3}j\sin(\frac{\pi}{3}) = 1 + \sqrt{3}\frac{1}{2} + j\sqrt{3}\frac{\sqrt{3}}{2} = 1 + \frac{\sqrt{3}}{2} + j\frac{3}{2}$$

Οπότε

$$|z| = \sqrt{\left(\frac{\sqrt{3}+2}{2}\right)^2 + \frac{9}{4}} \simeq 2.394$$

και

$$\theta = \tan^{-1} \frac{\frac{3}{2}}{\frac{2+\sqrt{3}}{2}} = \tan^{-1} \frac{3}{2+\sqrt{3}} = 0.677 \text{ rad}$$

Άρα

$$x(t) = 2.394e^{j(2\pi f_0 t + 0.677)}$$

Άσκηση 7

Έχουμε $z^6 = -1 \Leftrightarrow (\rho e^{j\theta})^6 = e^{j(2k\pi+\pi)} \Leftrightarrow \rho^6 e^{j6\theta} = e^{j(2k\pi+\pi)} \Leftrightarrow \left\{\begin{matrix}\rho^6 = 1 \\ 6\theta = 2k\pi + \pi\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\rho = |z| = 1 \\ \theta = \frac{2k\pi+\pi}{6}, \quad k = 0, 1, ..., 5\end{matrix}\right\}$

Άρα οι ρίζες είναι

$$z = \quad e^{j\frac{\pi}{6}}, \quad e^{j\frac{\pi}{2}}, \quad e^{j\frac{5\pi}{6}}, \quad e^{j\frac{7\pi}{6}}, \quad e^{j\frac{3\pi}{2}}, \quad e^{j\frac{11\pi}{6}}$$

Άσκηση 8

Κώδικας Matlab.