HY-215

Λύσεις 2ης σειράς ασκήσεων

1.

$$\begin{aligned}\int_{-T/2}^{T/2} x(t)z(t)dt &= \int_{-T/2}^{0} x(t)z(t)dt + \int_{0}^{T/2} x(t)z(t)dt \\ &= \int_{0}^{T/2} x(-t)z(-t)dt + \int_{0}^{T/2} x(t)z(t)dt \\ &= \int_{0}^{T/2} [x(-t)z(-t) + x(t)z(t)]dt = 0\end{aligned}$$

Υποθέσαμε οτι $x(-t) = x(t)$ και $z(-t) = -z(t)$

2.

$$\begin{aligned}x(t) = \frac{1}{2}\frac{e^{j2t} - e^{-j2t} + e^{j3t} - e^{-j3t}}{e^{jt} - e^{-jt}} &= \frac{1}{2}\frac{(e^{jt} - e^{-jt})[(e^{jt} - e^{-jt}) + 1 + (e^{jt} - e^{-jt})]}{e^{jt} - e^{-jt}} \\ &= \frac{1}{2} + \cos(t) + \cos(2t)\end{aligned}$$

$$T_1 = 2\pi$$
$$T_2 = \pi$$

Άρα

$$T_0 = 2\pi$$

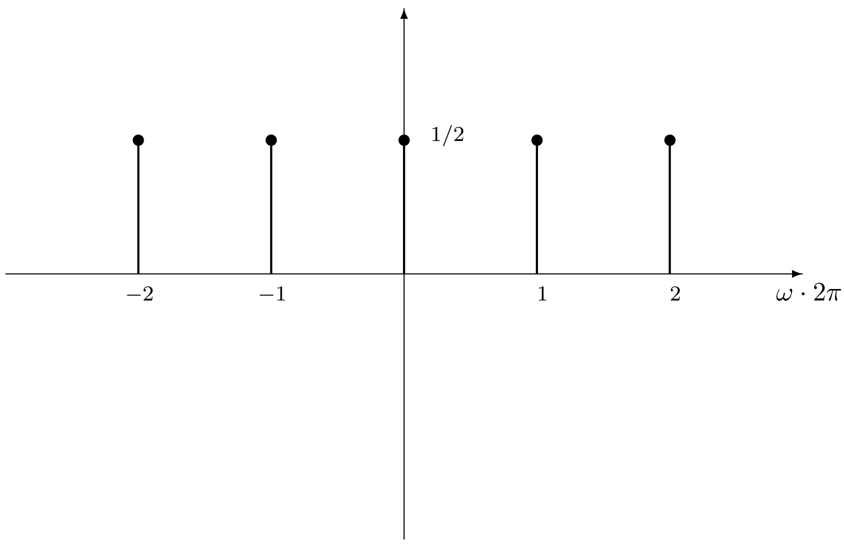

Figure 1:

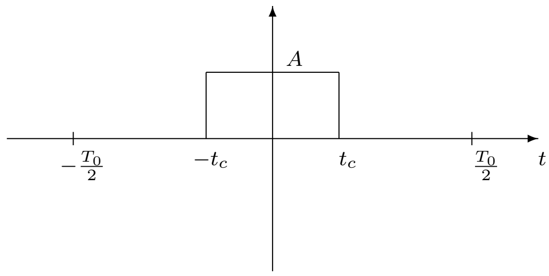

Figure 2:

3. Στη γενική μορφή (Σχ. 2)

$$X_0 = \frac{A}{T_0}\int_{-t_c}^{t_c} dt = A2\frac{t_c}{T_0}$$

$$X_k = \frac{A}{T_0}\int_{-t_c}^{t_c} e^{-jk2\pi f_0 t}dt = \frac{A}{T_0}\frac{1}{-jk2\pi f_0}\quad e^{-jk2\pi f_0}\Big|_{-t_c}^{t_c} = -\frac{1}{jk2\pi}\left(e^{-jk2\pi\frac{t_c}{T_0}} - e^{jk2\pi\frac{t_c}{T_0}}\right) =$$

$$= \frac{1}{k\pi}\sin(k2\pi\frac{t_c}{T_0})$$

Το σήμα $x(t)$ μπορεί να γραφτεί ως άθροισμα δυο σημάτων (Σχ. 3)

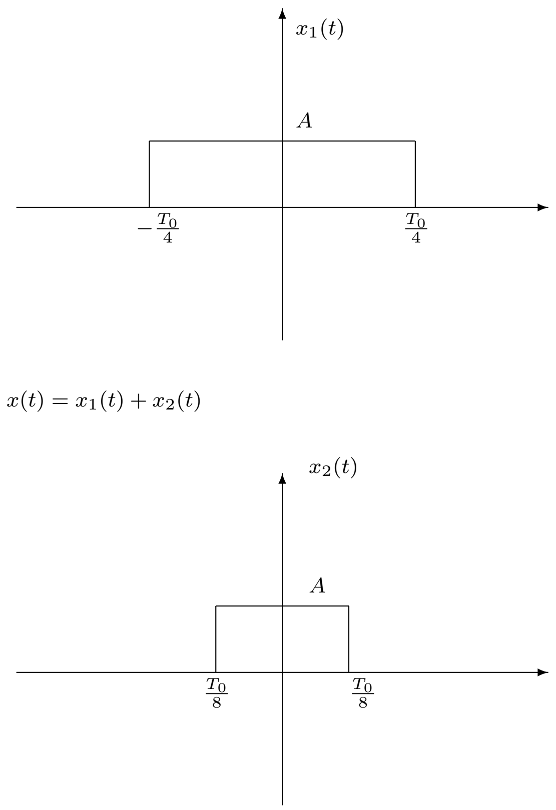

Figure 3:

$$x(t) = x_1(t) + x_2(t)$$

Για το $x_1(t)$

$$X_0 = \frac{A}{2} \qquad X_k = \frac{A}{k\pi}\sin(k\frac{\pi}{2})$$

Για το $x_2(t)$

$$\Psi_0 = \frac{A}{4} \qquad \Psi_k = \frac{A}{k\pi}\sin(k\frac{\pi}{4})$$

Άρα

$$x(t) = \frac{3A}{4} + 2\sum_{k=1}^{\infty}\frac{A}{k\pi}\left[\sin(k\frac{\pi}{2}) + \sin(k\frac{\pi}{4})\right]\cos(k2\pi f_0 t)$$

4.

$$w_1 = e^{-t} \qquad \|w_1(t)\|^2 = \int_0^{\infty} e^{-2t}dt = \frac{1}{2} \Rightarrow \|w_1(t)\| = \frac{1}{\sqrt{2}}$$

Άρα

$$\boxed{\psi_1(t) = \sqrt{2}e^{-t}}$$

$$w_2(t) = v_2(t) - \langle\, v_2(t)\psi_1^*(t)\,\rangle\, \psi_1(t) = e^{-2t} - \int_0^{\infty} e^{-2t}\sqrt{2}e^{-t}dt\sqrt{2}e^{-t} = e^{-2t} - \frac{2}{3}e^{-t}$$

$$\|w_2(t)\|^2 = \int_0^{\infty}(e^{-2t} - \frac{2}{3}e^{-t})^2 dt = \int_0^{\infty}(e^{-4t} - \frac{4}{3}e^{-3t} + \frac{4}{9}e^{-2t})dt = \frac{1}{36} \Rightarrow$$

$$\|w_2(t)\| = \frac{1}{6}$$

Άρα

$$\psi_2(t) = \frac{w_2(t)}{\|w_2(t)\|} \Rightarrow \boxed{\psi_2(t) = 6e^{-2t} - 4e^{-t}}$$

$$w_3(t) = v_3(t) - \Big(\langle\, v_3(t)\psi_1^*(t)\,\rangle\, \psi_1(t) + \langle\, v_3(t)\psi_2^*(t)\,\rangle\, \psi_2(t)\Big) =$$

$$= e^{-3t}\left[\int_0^{\infty} e^{-3t}\sqrt{2}\, e^{-t}dt\, \sqrt{2}e^{-t} + \int_0^{\infty} e^{-3t}(6e^{-2t} - 4e^{-t})\, dt\, (6e^{-2t} - 4e^{-t})\right] =$$

$$= e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}$$

$$\|w_3(t)\|^2 = \int_0^{\infty}(e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t})^2\, dt \; = \frac{1}{\sqrt{6}}\frac{1}{10}$$

Άρα

$$\boxed{\psi_3(t) = \sqrt{6}\Big(10e^{-3t} - 12e^{-2t} + 3e^{-t}\Big)}$$

5.

$$a_k = \int_0^{T} x(t)\psi_k^*(t)dt$$

Άρα

$$a_1 = \int_0^{T/2}\psi_1^*(t)dt \; - \int_{T/2}^{T}\psi_1^*(t)dt = \sqrt{2}\left[\int_0^{T/2} e^{-t}\, dt \; - \int_{T/2}^{T} e^{-t}\, dt\right] =$$

$$= \sqrt{2}\Big(1 - e^{-\frac{T}{2}} - e^{-T} + e^{-\frac{T}{2}}\Big) = \sqrt{2}\,(1 - e^{-T})$$

$$a_2 = \int_0^{T/2}(6e^{-2t} - 4e^{-t})\, dt - \int_{T/2}^{T}(6e^{-2t} - 4e^{-t})\, dt = 4e^{-T} - 3e^{-2T} - 1$$

$$a_3 = \sqrt{6}\left[10\int_0^{T/2} e^{-3t}dt - 12\int_0^{T/2} e^{-2t}dt + 3\int_0^{T/2} e^{-t}dt - 10\int_{T/2}^{T} e^{-3t}dt + 12\int_{T/2}^{T} e^{-2t}dt - 3\int_{T/2}^{T} e^{-t}dt\right] =$$

$$= \sqrt{6}(\frac{10}{3}e^{-3T} - 6e^{-2T} + 15e^{-T} - \frac{20}{3}e^{-3\frac{T}{2}} - 6e^{-\frac{T}{2}} + \frac{1}{3})$$

6. π.χ.

$$\int_0^\infty \psi_1(t)\psi_1^*(t)dt = 2\int_0^\infty e^{-2t}dt = 1$$

$$\int_0^\infty \psi_1(t)\psi_2^*(t)dt = \int_0^\infty \sqrt{2}e^{-t}(6e^{-2t} - 4e^{-t})\,dt = 2\sqrt{2}\int_0^\infty (3e^{-3t} - 2e^{-2t})\,dt =$$

$$2\sqrt{2}(3\frac{1}{3} - 2\frac{1}{2}) = 0$$

κ.λ.π

7.

$$\int_0^T w_k(t)w_l^*(t)\,dt = \int_0^T \sin(\frac{2\pi}{T}kt)\sin(\frac{2\pi}{T}\,l\,t)dt =$$

$$= \int_0^T \frac{1}{2}\sin(\frac{2\pi}{T}(k-l)t)\,dt\; - \;\frac{1}{2}\underbrace{\int_0^T \sin(\frac{2\pi}{T}(k+l)t)\,dt}_{0} =$$

$$= \frac{1}{2}\int_0^T \sin\left(\frac{2\pi}{T}(k-l)t\right)dt = \begin{cases} 0 & k \neq l \\ \frac{T}{2} & k = l \end{cases}$$

$$\|w_k(t)\|^2 = \frac{T}{2} \Rightarrow \|w_k(t)\| = \sqrt{\frac{T}{2}}$$

Άρα το ορθοκανονικό σύνολο είναι :

$$\boxed{\psi_k(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{2\pi}{T}kt\right)}$$