ΗΥ215: Λύσεις $4^{ης}$ Σειράς Ασκήσεων

1. Επειδή τα σήματα είναι πραγματικά έχουμε:

   $X_k = X^{\star}_{-k}$ και $Y_k = Y^{\star}_{-k}$

$$\sum_{k=-\infty}^{\infty} X_k\,\Psi_k^{\star} = \sum_{k=-\infty}^{-1} X_k\,\Psi_k^{\star} + \underbrace{X_0\,\Psi_0^{\star}}_{\text{μέση τιμή } 0} + \sum_{k=1}^{\infty} X_k\,\Psi_k^{\star} =$$

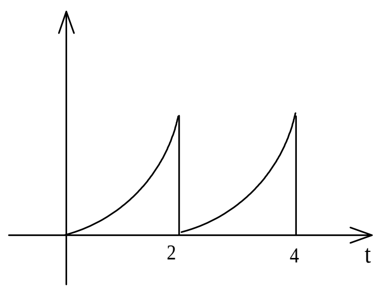
$$= \sum_{k=1}^{\infty} X_{-k}\,\Psi_{-k}^{\star} + \sum_{k=1}^{\infty} X_k\,\Psi_k^{\star} =$$

$$= \sum_{k=1}^{\infty} X_k^{\star}\,\Psi_k + \sum_{k=1}^{\infty} X_k\,\Psi_k^{\star} =$$

$$= \sum_{k=1}^{\infty} X_k\,\Psi_k^{\star} + (X_k\,\Psi_k^{\star})^{\star} =$$

$$= 2\sum_{k=1}^{\infty} \Re\{X_k\,\Psi_k^{\star}\}$$

2. $X_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{4}\,t^2\,dt = \frac{1}{2\pi}\,\frac{1}{4}\,\left.\frac{t^3}{3}\right|_0^{2\pi} = \frac{1}{3}\,\pi^2$



$$X_k = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{4}\,t^2\,e^{-jkt}\,dt = \frac{1}{8\pi}\left\{-\frac{1}{jk}\left[4\pi^2 + \frac{2}{jk}\left(2\pi + \frac{1}{jk}\cdot 0\right)\right]\right\} =$$

$$= \frac{1}{8\pi}\left\{-\frac{1}{jk}\left[4\pi^2 + \frac{4\pi}{jk}\right]\right\} = \frac{1}{2k^2} + \frac{\pi}{2k}\,e^{j\frac{\pi}{2}}$$

άρα

$$x(t) = \frac{1}{3}\pi^2 + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos(kt) - \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\sin(kt)$$

$$E = \frac{1}{2\pi}\int_0^{2\pi} x^2(t)\,dt = \frac{\pi^4}{5}$$

$-10:10 \rightarrow 19.020$ δηλ. 97.59% της ενέργειας $\sum_{|k|>10}|X_k|^2 = 0.4698$

$-20:20 \rightarrow 19.244$ δηλ. 98.76% της ενέργειας $\sum_{|k|>20}|X_k|^2 = 0.24$

3. $d^2(x,y) = \int_{t_1}^{t_2}|x(t)-y(t)|^2\,dt = \int_{t_1} t_2(x(t)-y(t))\,(x^\star(t)-y^\star(t))\,dt =$

$$= \int_{t_1}^{t_2} x(t)\,x^\star(t)\,dt + \int_{t_1}^{t_2} y(t)\,y\star(t)\,dt - \int_{t_1}^{t_2} x(t)\,y\star(t)\,dt - \int_{t_1}^{t_2} x^\star(t)\,y(t)\,dt =$$

$$= ||x||^2 + ||y||^2 - \int_{t_1}^{t_2}[(x(t)\,y^\star(t)) + (x(t)\,y^\star(t))^\star]\,dt =$$

$$= ||x||^2 + ||y||^2 - 2\,\Re\{\int_{t_1}^{t_2} x(t)\,y^\star(t)\,dt\} =$$

$$= ||x||^2 + ||y||^2 - 2\,\Re < x, y >$$

Προφανώς η μέγιστη απόσταση συμβαίνει όταν $x \perp y \Rightarrow < x, y^\star >= 0 \Rightarrow d^2(x,y) = ||x||^2 + ||y||^2$
Όταν $x \parallel y$ π.χ. $x(t) = k\,Y(t) \Rightarrow ||x||^2 = k^2\,||y||^2$
Άρα τότε $d^2(x,y) = k^2\,||y||^2 + ||y||^2 - 2k||y||^2 = ||y||^2\,(1-k)^2$

4. $d^2(x,y) = ||x||^2 + ||y||^2 - \frac{2}{T}\,\Re < x, y\star >$

$||x||^2 = \frac{1}{T}\int\limits_0^T x^2(t)\,dt = \frac{A^2}{2}$ από το θεώρημα Parseval

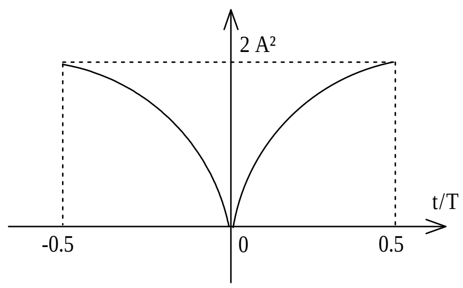
$||y||^2 = \frac{A^2}{2}$ (παρομοίως)

$-\frac{2}{T}\,\Re < x, y^\star > = -\frac{2}{T}\int\limits_0^T x(t)\,y(t)\,dt = -\frac{2}{T}\,A^2\int\limits_0^T \cos(\omega_0 t)\,(\cos(\omega_0 t)\,\cos(\omega_0\tau) + \sin(\omega_0 t)\,\sin(\omega_0\tau)) =$

$= -\frac{2}{T}\,A^2\,[\underbrace{\int\limits_0^T \frac{1}{2}\,\cos(2\omega_0 t)\,dt}_{0} + \int\limits_0^T \frac{1}{2}\,\cos(\omega_0\tau)\,dt + \underbrace{\int\limits_0^T \cos(\omega_0 t)\,\sin(\omega_0 t)\,dt}_{0}\cdot\sin(\omega_0\tau)] =$

$= -A^2\,\cos(\omega_0\tau)$

Άρα: $d^2(x,y) = A^2 - A^2\,\cos(\omega_0 t)$



5. Θα πρέπει να δείξουμε ότι $\int\limits_0 T\psi_k(t)\,\psi_l(t)\,dt = 0$, για $k \neq l$

$\int\limits_0^T \psi_1(t)\,\psi_2(t)\,dt = \int\limits_0 \frac{T}{2}\,dt - \int\limits_{\frac{T}{2}}^T dt = \frac{T}{2} - (T - \frac{T}{2}) = 0$

$\int\limits_0^T \psi_1(t)\,\psi_3(t)\,dt = -\int\limits_0 \frac{T}{4}\,dt + \int\limits_{\frac{T}{4}}^{\frac{3T}{4}} dt - \int\limits_{\frac{3T}{4}}^T dt = -\frac{T}{4} + \frac{3T}{4} - \frac{T}{4} - (T - \frac{3T}{4}) = 0$

$\int\limits_0^T \psi_2(t)\,\psi_3(t)\,dt = -\int\limits_0 \frac{T}{4}\,dt + \int\limits_{\frac{T}{4}}^{\frac{T}{2}} dt - \int\limits_{\frac{T}{2}}^{\frac{3T}{4}} dt + \int\limits_{\frac{3T}{4}}^T dt = -\frac{T}{4} + \frac{T}{2} - \frac{T}{4} - (\frac{3T}{4} - \frac{T}{2}) + T - \frac{3T}{4} = 0$

6. $\lambda_{kl} = < \psi_k, \psi_l^\star > = \int\limits_0^\infty e^{-(k+l)t}\, dt = \frac{1}{k+l},\ k, l = 1, 2, 3$

$\gamma_l = < x, \psi_l^\star > = \int\limits_0^\infty x(t)\, e^{-lt}\, dt = \int\limits_0^1 e^{-lt}\, dt = \frac{1}{l}\,[1 - e^{-l}]$

Έτσι:

$$\left.\begin{array}{l} \Lambda = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \Rightarrow \Lambda^{-1} = \begin{bmatrix} 72 & -240 & 180 \\ -240 & 400 & -720 \\ 180 & -720 & 600 \end{bmatrix} \\ \Gamma = \begin{bmatrix} 0.63212 \\ 0.43233 \\ 0.31674 \end{bmatrix} \end{array}\right\} \Rightarrow$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \Lambda^{-1} \cdot \Gamma = \begin{bmatrix} -1.234 \\ 9.338 \\ -7454 \end{bmatrix} \Rightarrow \hat{x}(t) = -1.234\, e^{-t} + 9.338\, e^{-2t} - 7.454\, e^{-3t}$$