COMMUNICATION OVER FADING DISPERSIVE CHANNELS, OPTIMAL RECEIVERS AND SIGNALS

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Abstract. We study the problem of designing the optimal receiver for a dispersive channel and the optimal signals to be transmitted over a given dispersive channel. The assumed transmission channel is WSSUS (Wide Sense Stationary Uncorrelated Scatterers), randomly time variant, so it is characterized by its scattering function. We study first the effective realization of the optimal receiver, given a scattering function, and we point out that the structure of the optimal receiver varies according to the discrete or continuous structure of the scattering function over the time frequency plane. We give some precise results in the case of multipath transmission and in the case of modulation-transmission.

In a second part, we are interested in the signal design problem. Some new results are given which permit defining—in a way more direct than usual—the class of optimal signals for a given dispersive communication.

Zusammenfassung. Wir studieren das Problem von optimalen Empfängern für dispersive Kanäle sowie von optimalen Signalen für einen gegebenen dispersiven Kanal. Es wird angenommen dass der gegebene Kanal WSSUS ist (Wide Sense Stationary Uncorrelated Scatterers), stochastisch zeitvariabel; er ist also durch seine 'scattering' Funktion charakterisiert. Wir studieren zuerst die effective Realisierung des optimalen Empfängers für eine gegebene 'scattering' Funktion, und wir zeigen dass sich die Struktur des optimalen Empfängers verändert, je nach Struktur der kontinuierlichen oder diskreten 'scattering' Funktion über der Zeit-Frequenz Ebene. Wir geben Resultate im Fall von 'multipath' Übertragung sowie im Fall von Modulations-Übertragung.

In der zweiten Hälfte analysieren wir das Problem des optimalen Signals. Neue Ergebnisse werden vorgestellt die es erlauben, in einer direkteren Art als bisher, die Klasse von optimalen Signalen für einen gegebenen dispersiven Kanal zu definieren.

Résumé. Nous étudions le problème de détermination du récepteur optimal après transmission dans un canal dispersif et des signaux optimaux à transmettre dans une situation donnée. La voie de transmission est modélisée par un filtre aléatoire linéaire variant au cours du temps et supposé WSSUS (stationnaire au sens large et aux diffuseurs non-corrélés). Ainsi, il est caractérisé par sa fonction de diffusion. Nous étudions premièrement la réalisation effective du récepteur optimal, pour une fonction de diffusion et un signal à l'émission donnés. Nous montrons que la structure du récepteur optimal est différente suivant la structure continue ou discrète de la fonction de diffusion sur le plan temps-fréquence. Nous donnons certains résultats précis dans le cas de la transmission multitrajets et dans le cas de la modulation aléatoire. Dans une seconde partie, nous nous intéressons au problème de choix des signaux à l'émission. Certains résultats nouveaux sont donnés qui permettent de définir directement la classe des signaux optimaux pour une communication donnée.

Keywords. Fading dispersive channel, detection, binary communication, optimal receiver, signal design.

1. Introduction

In many propagation channels (ionospheric, seismic, tropospheric, aerian or submarine acoustic channels, urban communication), in addition to the additive noise, the transmitted signal is randomly distorted in many ways. This is called fading transmission, taking into account both random frequency filtering and

temporal modulation. The theoretical model used to describe this phenomena is well known. It was first treated by Zadeh [22] and Kailath [9] and later, by Bello [1], Ellinthorpe [2], Kennedy [10], Laval [12], G. Jourdain [8]. This model is a randomly time variant filter, and with the WSSUS[1] or quasi WSSUS hypothesis, the second order characterization of the medium is its scattering function.

This paper has two aims: first, it presents some new results about the choice of the model adopted for the dispersive channel ('discrete' or 'continuous' structure of the scattering function, Section 3) and some new results about the signal design problem and optimal performance (Section 4). In addition, it assembles some other recent results of the authors, results which have been detailed in separate ways elsewhere, but that are recalled here in a comprehensive view. So this paper constitutes an extension of the classical results of Van Trees or Kennedy for the problem of optimal communication in a dispersive channel.

In the second Section, we recall the hypothesis of the problem which is the detection of a gaussian signal in gaussian noise. The theoretical structures have been given [19], but generally, it is nearly impossible to elaborate the optimal receiver and to calculate its performance. We give some new formulas (Section 2.3.2) for approximating the error probability, which exhibit well the parameters of the dispersive channel.

In the third Section, we propose some structures of optimal receivers associated with particular forms of the scattering function. We distinguish the cases where, with a 'discrete' scattering function, an internal model of the medium and of the optimal receiver is not necessary; and the opposite case where an internal model is necessary. This technique was originally studied by Kurth [11]. In particular a new procedure—the so-called 'factorable covariance'—enables us to solve the problem directly when the scattering function is constituted by a set of Dirac functions in the time and frequency planes. This corresponds, for example, to the multipath transmission that is often encountered in some propagation channel—it has been illustrated here in acoustic submarine propagation. Some recent results are given for the performance of the optimal receiver after a multipath transmission, using an ideal or non ideal multipath model.

The fourth Section concerns the choice of signals to be used in a fading dispersive channel. It is well known that in this case, the form of the signal has a direct effect upon the performance of the receiver. Some recent calculations [15] have shown that the choice of the signal may be guided by means of its ambiguity function. We present new results about the variance of the sufficient statistic, and we give some ideas for choosing the optimal signal in the frequency dispersive channel.

2. Basic hypothesis and results about the communication in a dispersive channel

2.1. The transmission model

The deterministic signal $f(t)$, with energy E and duration T_0, is transmitted through the (assumed) random linear channel. The input and output signals are assumed—it is practically always true—bandpass around the center frequency ν_0. The transmission model is:

$$\tilde{s}(t) = \int \tilde{f}(t-\xi)\tilde{H}(t,\xi)\,\mathrm{d}\xi. \tag{1}$$

All symbols below, noted with $\tilde{\ }$, correspond to complex amplitudes relative to ν_0. For example, $\tilde{f}(t)$ and $\tilde{s}(t)$ are the complex amplitudes of the input and output signals, relative to ν_0, and $\tilde{H}(t,\xi)$ is the

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[1] Wide Sense Stationary Uncorrelated Scatterers.

complex random bitemporal response of the medium:

$$f(t) = \sqrt{2} \, \text{Re}\{\tilde{f}(t) \, e^{i2\pi\nu_0 t}\}; \quad s(t) = \sqrt{2} \, \text{Re}\{\tilde{s}(t) \, e^{i2\pi\nu_0 t}\}.$$

Then:

$$E = \int_0^{T_0} |f(t)|^2 \, dt = \int_0^{T_0} |\tilde{f}(t)|^2 \, dt. \tag{2}$$

The general statistical hypothesis is the following: $\tilde{H}(t, \xi)$ is a zero mean random function: this can often be verified practically. For the second order, we assume the channel is WSSUS (Wide Sense Stationary Uncorrelated Scatterers). This means that the covariance of $\tilde{H}(t, \xi)$ is stationary with respect to t and uncorrelated with respect to ξ:

$$E\{\tilde{H}(t, \xi)\tilde{H}^*(t', \xi')\} = \Gamma_H^1(t - t', \xi)\delta(\xi - \xi'). \tag{3}$$

This hypothesis can also be verified in some practical cases, at least for some bandwidth. In this case, the channel is statistically characterized by its scattering function $\tilde{S}(\nu, \xi)$, which is the Fourier Transform of Γ_H^1 with respect to $t - t'$.

$$\Gamma_H^1(t - t', \xi) \underset{t-t'}{\overset{\nu}{\rightleftharpoons}} \tilde{S}(\nu, \xi).$$

$\tilde{H}(t, \xi)$ is also assumed to be gaussian; so it is entirely characterized by its two first moments.

In this case, $\tilde{s}(t)$ is also a gaussian, complex, zero-mean process, with a nonstationary covariance given by:

$$\tilde{\Gamma}_s(t, u) = \iint \tilde{f}(t - \xi)\tilde{f}^*(u - \xi)\tilde{S}(\nu, \xi) \, e^{i2\pi\nu(t-u)} \, d\nu \, d\xi. \tag{4}$$

The received instantaneous power depends on the transmitted power and the scattering function:

$$E\{|\tilde{s}(t)|^2\} \triangleq \tilde{\Gamma}_{\tilde{s}}(t, t) = \iint \tilde{S}(\nu, \xi)|\tilde{f}(t - \xi)|^2 \, d\nu \, d\xi. \tag{5}$$

This last formula illustrates the phenomenon of transmitted power dispersion.

After transmission the signal $\tilde{s}(t)$ is corrupted by additive noise with complex amplitude $\tilde{n}(t)$. In order to facilitate the calculus this noise is also assumed to be gaussian, zero mean and white, with a power spectral density N_0.
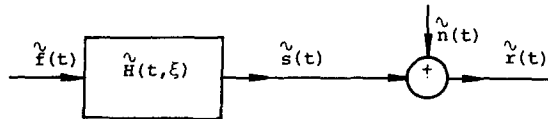

Fig. 1. The transmission model.

The received mean power $\overline{E_r}$ and the input receiver signal to noise ratio are defined by:

$$\frac{\overline{E_r}}{N_0} \triangleq \frac{1}{N_0} \int_0^T \tilde{\Gamma}_{\tilde{s}}(t, t) \, dt = \frac{E}{N_0} \iint_{R^2} \tilde{S}(\nu, \xi) \, d\nu \, d\xi \triangleq M\frac{E}{N_0}, \tag{6}$$

where T is the receiver observation interval ($T \geq T_0$).

2.2. Detection and communication. Optimal realizable receiver

Two kinds of problem are considered: on-off detection and binary communication. In the first problem, $\tilde{r}(t)$ is tested for the presence of $\tilde{s}(t)$:

$$H_1: \tilde{r}(t) = \tilde{s}(t) + \tilde{n}(t), \quad t \in [0, T];$$
$$H_0: \tilde{r}(t) = \tilde{n}(t), \quad t \in [0, T].$$
(7)

In the second problem, the choice needs to be made between $\tilde{s}_1(t)$ or $\tilde{s}_0(t)$:

$$H_1: \tilde{r}(t) = \tilde{s}_1(t) + \tilde{n}(t), \quad t \in [0, T];$$
$$H_0: \tilde{r}(t) = \tilde{s}_0(t) + \tilde{n}(t), \quad t \in [0, T].$$
(8)

It is then well known [20] that, for example in the case of (7) the bayesian optimal receiver must calculate the statistic l, that is a quadratic functional of the observation $\tilde{r}(t)$ and compare it to a threshold γ. This direct derivation leads to a noncausal filter.

$$l = \frac{1}{N_0} \int\int \tilde{r}^*(t) \tilde{h}(t, u) \tilde{r}(u) \, dt \, du \overset{H_1}{\underset{H_0}{\gtrless}} \gamma = \ln \eta + \sum_{i=1}^{\infty} \ln \left(1 + \frac{\lambda_i}{N_0}\right),$$
(9)

where (i) $\tilde{h}(t, u)$ satisfies the integral equation:

$$N_0 \tilde{h}(t, u) + \int_0^T \tilde{h}(t, z) \tilde{\Gamma}_s(z, u) \, dz = \tilde{\Gamma}_s(t, u) \quad 0 \leq t, u \leq T.$$
(10)

(ii) the λ_i are the eigenvalues of the covariance $\tilde{\Gamma}_s(t, u)$. The threshold η is calculated as a function of the a priori probabilities and the costs of the two hypotheses. If they are identical, $\eta = 1$.

The difficulty lies in the solution of the integral equation (9) that leads to an unrealizable filter $\tilde{h}(t, u)$.

The procedure for deriving the optimal realizable (causal) structure consists of the following steps: to derive from $\tilde{r}(t)$, an optimal estimate of $\tilde{s}(t)$, we call $\hat{\tilde{s}}_r(t)$, to correlate $\tilde{s}_r(t)$ with $\tilde{r}(t)$; and also to square it. The schema is given below:
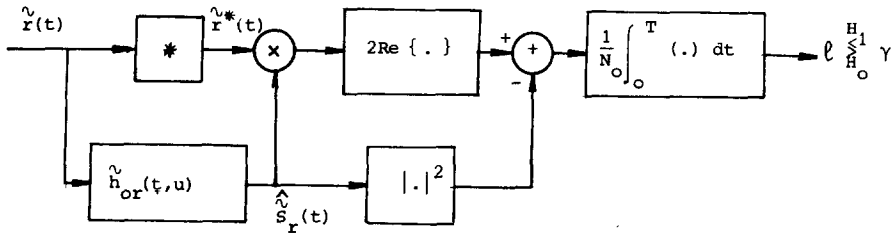

Fig. 2. The optimal realizable receiver.

The realizable filter $\tilde{h}_{or}(t, u)$ satisfies the following integral equation:

$$N_0 \tilde{h}_{or}(t, u) + \int_0^t \tilde{h}_{or}(t, z) \tilde{\Gamma}_{\tilde{s}}(z, u) \, dz = \tilde{\Gamma}_{\tilde{s}}(t, u), \quad 0 \leq u \leq t,$$
(11)

$$l = \frac{1}{N_0} \int_0^T [2 \operatorname{Re}\{\tilde{r}^*(t) \hat{\tilde{s}}_r(t)\} - |\hat{\tilde{s}}_r(t)|^2] \, dt.$$
(12)

2.2.1. Particular cases

The filtering of $\tilde{r}(t)$ by $\tilde{h}_{or}(t, u)$ implies the solution of the Wiener equation (11). This is ordinarily impossible except for some very particular covariances $\tilde{\Gamma}_{\tilde{s}}$. Van Trees has exhibited three limit cases in which equation (11) is simplified.

(a) If the covariance $\tilde{\Gamma}_{\tilde{s}}$ can be considered to be stationary and the observation time T is much larger than the characteristic evolution time of the signal covariance.

(b) If the number of the covariance eigenvalues and eigenfunctions is limited. But one must still calculate them!

(c) If the ratio of the signal to noise eigenvalues is low.

Another case leads to the direct solution of (9)—the so-called 'factorable covariance' case—and is studied in the third Section: By these means some different practically encountered cases can be solved.

2.2.2. State-variable model

More generally, in order to obtain $\hat{\tilde{s}}_r(t)$, the elaboration of a state-variable model for $\tilde{s}(t)$ is needed. The optimal receiver is then determined by means of a state-variable model: that is the optimal filter $\tilde{h}_{or}(t, u)$ of Fig. 2 is replaced by a Kalman filter. This kind of structure has been developed by Kurth [11]. We have also worked on this structure for different applications.

The first choice is to represent the channel itself (i.e. $\tilde{H}(t, \xi)$) by means of a state-variable model. Thus, when the emitted signal $\tilde{s}(t)$ changes, the channel model remains valid. Second, it is possible to show [11] that, if the scattering function $\tilde{S}(\nu, \xi)$ is rational in terms of ν, the state-variable model of $\tilde{H}(t, \xi)$—and consequently this of $\tilde{s}(t)$—is easily derived. The general structure of the state-variable model of $\tilde{H}$ and $\tilde{s}$ is given in the Annex, where the state-variable model of the optimal estimate $\hat{\tilde{s}}_r(t)$ can also be found.

This state-variable model is relative to the variable t in $\tilde{H}(t, \xi)$ and consequently the variable ν in $\tilde{S}(\nu, \xi)$, while ξ is considered as a parameter. The main difficulty in realizing this kind of estimate comes from the requirement for ξ-parametrization. The solution becomes realistic when $\tilde{S}(\nu, \xi)$ is quantized versus ξ: it will be seen later in the third Section.

Remark. In the case of binary communication, two identical 'branches' similar to that of Fig. 2 must be elaborated—one for $\tilde{s}_1(t)$, one for $\tilde{s}_0(t)$—and the two statistics l_1 and l_0 must be compared.

2.3. Performance

In order to know the quality of the optimal receiver and to eventually compare it to suboptimal receivers, for example in problem (7), the detection probability P_D and the false alarm probability P_F are usually used:

$$P_D = \int_\gamma^\infty P_{l/H_1}(x)\, \mathrm{d}x; \qquad P_F = \int_\gamma^\infty P_{l/H_0}(x)\, \mathrm{d}x, \tag{13}$$

where the probability density function (pdf) of l under the two hypotheses are needed. Generally they are very difficult to calculate because of the quadratic form of the receiver.

In problem (8), an error probability is calculated:

$$P(e) = \tfrac{1}{2}[P(H_1/H_0) + P(H_0/H_1)], \tag{13 bis}$$

by assuming the two hypotheses equally likely.

2.3.1. Classical results

(a) *In the on-off case.* Usually, some approximate formulas given by [3] are used in this case. These formulas are based upon a development that uses the log of the moment generating function of the statistic l under the H_0 hypothesis.

$$\mu(s) = \log E\{\exp(sl(r))|H_0\}. \tag{14}$$

In this quantity, s is a variable depending upon the threshold (if $\eta = 1$ in the equation (9), s is chosen so that $\dot{\mu}(s) = 0$).

The calculation of $\mu(s)$ is different according to whether a state variable model is used, or not. Using $\mu(s)$, various approximate expressions for P_D and P_F are calculated [19].

(b) *In the binary communication case.* Some results have been given for the general case [19]. In the following we are concerned with the symmetrical case only: that is, $\tilde{s}_0(t)$ and $\tilde{s}_1(t)$ have identical eigenvalues (and the same energy) and their eigenfunctions are mutually orthogonal.[2]

It is known that the error probability is then bounded by:

$$P(e) \leqslant \tfrac{1}{2} \cdot \exp\left(\mu(1/2)\frac{\overline{E_r}}{N_0}\right). \tag{15}$$

The quantity $\mu(1/2)$ is often considered to be an efficiency measure of a particular communication system.

2.3.2. New results

We have proposed [15] new approximate formulas for the performance calculation. They have a twofold advantage:

(i) They connect receiver performance directly to the *emitted* signal $\tilde{f}(t)$ and not to the *received* signal $\tilde{s}(t)$: this is a major point in the signal design problem (cf. Section 4).

(ii) They are well adapted when the scattering function exhibits a discrete structure (cf. below Section 3.1.2).

These formulas are given below:

(a) in the on-off case:

$$P_D = 1 - I_\Gamma\left(\sqrt{a+1} \cdot \frac{\ln(1+b)}{b}, a\right),$$

$$P_F = 1 - I_\Gamma\left(\sqrt{a+1} \cdot \frac{(1+b) \cdot \ln(1+b)}{b}, a\right), \tag{18}$$

where I_Γ is the incomplete Gamma function (Pearson's form [14]), a and b are the only parameters of these approximate formulas and are connected to the second order moments of l under the H_1 hypothesis:

$$a+1 = \frac{(E\{l|H_1\})^2}{\text{var}\{l|H_1\}}, \qquad b = \frac{\text{var}\{l|H_1\}}{E\{l|H_1\}}. \tag{19}$$

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[2] Let us illustrate this case by a simple example: $\tilde{f}_0(t)$ and $\tilde{f}_1(t)$ are assumed orthogonal (ex FSK) and the channel is a two path channel (cf. Section 3.1.2.b) with delay L. The orthogonality condition is verified if the correlation $c_{f_0f_1}(\tau)$ between $\tilde{f}_0$ and $\tilde{f}_1$ is zero for $\tau = 0$, $+L$ and $-L$.

They may be calculated with (9) and (10), and with (6):

$$E\{l|H_1\} = \frac{\overline{E_r}}{N_0},\tag{20}$$

$$\text{var}\{l|H_1\} = \frac{1}{N_0^2} \int \int |\tilde{\Gamma}_{\tilde{s}}(t, u)|^2 \, dt \, du.\tag{21}$$

The quantity $a + 1$ can be interpreted as a 'diversity-path effective number', and b as a signal to noise ratio by path.

(b) In the binary symmetrical communication case.

The same reference [15] gives a new formula for the error probability $P(e)$:

$$P(e) = \frac{\Gamma(2a+2)}{\Gamma(a+1)\Gamma(a+2)} \left(\frac{1+b}{(2+b)^2}\right)^{a+1} {}_2F_1\left(2a+2, 1, a+2; \frac{1}{2+b}\right),\tag{22}$$

where ${}_2F_1$ is the hypergeometric function [25], Γ is the gamma function and a and b are always given by the same formula (19) in which the statistic is now the output l_1 of the branch corresponding to $\tilde{s}_1$.

If a is an integer $(a = m)$ one can recover the formula known in the orthogonal signals diversity cases [13]:

$$P(\varepsilon) = \left(\frac{1}{2+b}\right)^{m+1} \sum_{n=0}^{m} \binom{m+n}{n} \left(\frac{1+b}{2+b}\right)^m.\tag{23}$$

This is consistent with the interpretation of parameters a and b given above.

3. Optimal receiver—structure and performance

Different cases are to be considered in this Section, related to different classes of the channel scattering function. The optimal receiver structure depends on whether the scattering function exhibits a fully discrete form in the time frequency plane, or not.

3.1. Fully discrete scattering function ('factorable' covariance). Example of multipath case

3.1.1. Optimal receiver in the factorable case

The optimal receiver structure is related directly to the covariance of the output signal $\tilde{s}(t)$ (see equations (6) and (7)). In addition to the three limit cases mentioned above, it is possible [17] to obtain an exact solution to equation (6) if the covariance of $\tilde{s}(t)$ can be 'factorized' as below:

$$\tilde{\Gamma}_{\tilde{s}}(t, u) = \tilde{\psi}^\dagger(t) A \tilde{\psi}(u), \quad 0 \le t, u \le T\tag{24}$$

where $\tilde{\psi}(t)$ is a function vector and A is a constant hermitian matrix. This case (24) corresponds to a generalization of the 'separable kernel case' (case (ii), above mentioned). Effectively if (24) is substituted in (10):

$$N_0 \tilde{h}(t, u) + \int_0^T \tilde{h}(t, z) \tilde{\psi}^\dagger(z) A \tilde{\psi}(u) \, dz = \tilde{\psi}^\dagger(t) A \tilde{\psi}(u).$$

The solution $\tilde{h}(t, u)$ has a similar form:

$$\tilde{h}(t, u) = \tilde{\psi}^{\dagger}(t) \boldsymbol{B} \tilde{\psi}(u),$$

$$\boldsymbol{B} = \boldsymbol{A}(N_0 \mathbb{1} + \boldsymbol{CA})^{-1}.$$

$\boldsymbol{C}$ measures the correlation between the functions $\tilde{\psi}(t)$.

$$\boldsymbol{C} = \int_0^T \tilde{\psi}(z) \tilde{\psi}^{\dagger}(z) \, \mathrm{d}z.$$

The matrix $(N_0 \mathbb{1} + \boldsymbol{CA})$ is always invertible, for the eigenvalues of $\tilde{\Gamma}_s$ are the same as those of $\boldsymbol{CA}$ matrix.

It can be deduced that the optimal receiver computes the statistic l given by:

$$l = \tilde{r}^{\dagger} \frac{1}{N_0} \boldsymbol{B} \tilde{r}, \tag{25}$$

where:

$$\tilde{r} = \int_0^T \tilde{r}(t) \tilde{\psi}(t) \, \mathrm{d}t.$$

Thus, a new realizable structure of the optimal receiver is obtained without a state variable model of the medium. The scheme below illustrates this structure:
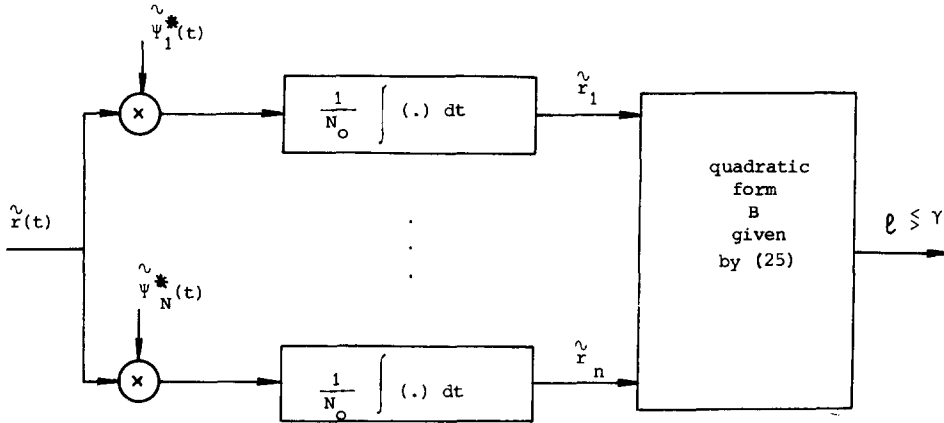

Fig. 3. The optimal receiver in the factorizable case.

If the matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ are diagonal (then the functions $\tilde{\psi}_i(t)$ are orthogonal but they may have unequal energy), $\boldsymbol{B}$ is itself diagonal and the quadratic form (25) is greatly simplified.

Let us see now the practical cases corresponding to the hypothesis (24).

3.1.2. Ideal multipath transmission

Multipath propagation is very often encountered in ionospheric transmission, or urban communication [21] and we have seen it in submarine acoustics [5]. The received signal is then the sum of several signals propagated through N paths:

$$\tilde{s}(t) = \sum_{i=1}^{N} \tilde{\alpha}_i \tilde{f}(t - \xi_i). \tag{26}$$

The complex amplitudes $\tilde{\alpha}_i$ along each path are assumed to be independent zero mean complex gaussian variables, with respective variance $q_i = E\{|\tilde{\alpha}_i|^2\}$. This corresponds to a Rayleigh fading over each path. The corresponding scattering function is discrete and exists only over the delay axis:

$$\tilde{S}(\nu, \xi) = \sum_{i=1}^{N} q_i \delta(\xi - \xi_i)\delta(\nu). \tag{27}$$

Thus, substituting (27) in (4) the covariance of $\tilde{s}(t)$ is,

$$\tilde{\Gamma}_{\tilde{s}}(t, u) = \sum_{i=1}^{N} q_i \tilde{f}(t - \xi_i)\tilde{f}^*(u - \xi_i), \tag{28}$$

which is of the form (24) with a diagonal matrix A and a vector $\tilde{\psi}$ given below:

$$A = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_N \end{bmatrix}; \qquad \tilde{\psi}(t) = \begin{bmatrix} \tilde{f}(t - \xi_1) \\ \cdot \\ \tilde{f}(t - \xi_N) \end{bmatrix}.$$

The optimal receiver is given by (25) in which the correlation matrix C consists of the correlation coefficients of the emitted $\tilde{f}(t)$ calculated for the path delays:

$$C_{ij} = \int_0^T \tilde{f}(t - \xi_i)\tilde{f}^*(t - \xi_j)\, dt \triangleq E \cdot \rho_{ij}.$$

The matrix B that defines the quadratic combination is:

$$B = \frac{1}{N_0} A \left[\mathbb{1} + \frac{E}{N_0} RA \right]^{-1},$$

in which R is the matrix of the coefficients ρ_{ij}.

(a) If the correlation time of $f(t)$ is less than the minimal delay between the paths, the C matrix is diagonal and so is B.

$$B = \frac{1}{N_0} \begin{bmatrix} q_1 \Big/ \left(1 + \dfrac{E}{N_0} q_1\right) & & 0 \\ & \ddots & \\ 0 & & q_N \Big/ \left(1 + \dfrac{E}{N_0} q_N\right) \end{bmatrix}$$

$\bar{E}_i = (E/N_0)q_i$ is the signal to noise ratio along the ith path. So in this case, the optimal receiver has N 'branches' (in which $\tilde{r}(t)$ is correlated with $\tilde{f}(t - \xi_i)$ and weighted by $\bar{E}_i/(1 + \bar{E}_i)$ before summation (see Fig. 4). The N correlators can be replaced by one filter matched to the signal $f(t)$ and followed by N delay lines ξ_i. If the signal to noise ratios over each path are nearly identical, the weights can disappear.

(b) The two-path case has been completely studied in [7] for example, and can be extended to N paths.
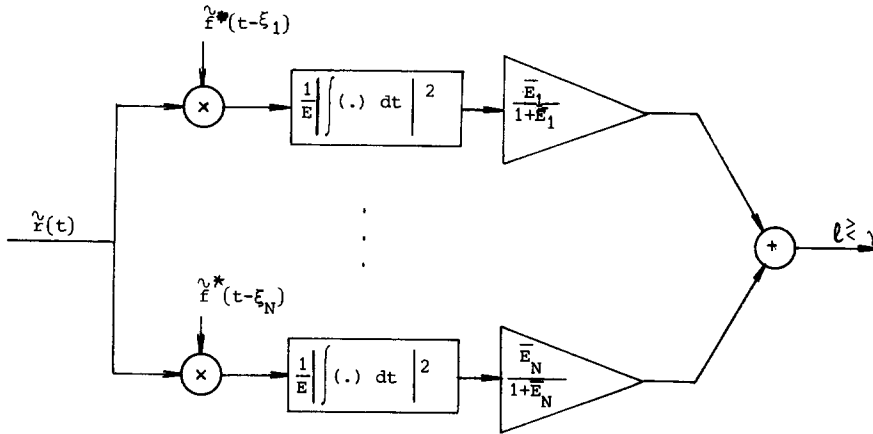
The calculation of receiver performance is given there. In the two-path case, the only parameters of the medium are:

· the delay L between the two paths which is then supposed known with the precision of the inverse signal bandwidth

· q_1/q_2 the ratio of the mean power over the two paths

· the correlation coefficient:

$$\rho = \frac{1}{E} \int_0^T \tilde{f}(t)\tilde{f}^*(t - L)\, dt, \quad T = T_0 + L. \tag{29}$$

Fig. 4. Multipath optimal receiver with B diagonal.

We present here only the results of the $P_D(P_F)$ variations, versus a given received signal to noise ratio (equal to $1, 2, \ldots, 50$) (see Fig. 5). These curves correspond to the $\rho = 0$ case, which leads to the best performance. We have also presented the case of a sub-optimal receiver using only one path.

When the SNR is greater than about 3, reception is better if the transmitted power is identically distributed over the two paths ($q_1 = q_2$). This is equivalent to a second order temporal diversity [13]. When the SNR is less than 3, the results depend from the threshold and the SNR ratio for the two paths.
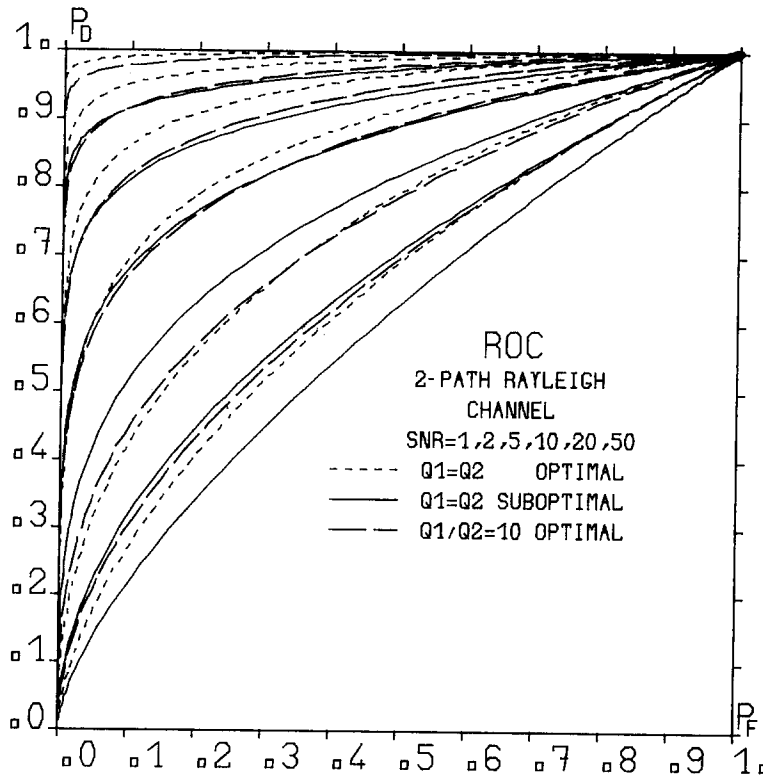
Fig. 5. Performance of the optimal receiver in the two-path case ($\rho = 0$).

In the 'equal energy and uncorrelated' case ($q_1 = q_2$; $\rho = 0$) we give here the P_D and P_F expressions calculated directly:

$$P_D = \left(1 + \frac{\eta}{(qE/N_0)}\right) e^{-\eta/(qE/N_0)},$$

$$P_F = \left[1 + \eta \frac{(1 + qE/N_0)}{qE/N_0}\right] e^{-\eta(1+qE/N_0)/(qE/N_0)}. \tag{30}$$

One can see that the approximate formula (18) given in Section 1 by making $q_1 = q_2 = q$ in (28), becomes the exact formula (30) above: in fact, in the two-path case, by substituting (28) into (21) one obtains:

$$\iint |\tilde{\Gamma}_s(t, u)|^2 \, dt \, du = 2q^2 E^2.$$

The parameters a and b can then be calculated by means of (19), (20) and (21). One obtains:

$$a + 1 = 2, \qquad b = \frac{\bar{E}_r}{2N_0}. \tag{31}$$

These two results confirm the interpretation given in Section 2.3.2. If the expressions (31) are substituted into (18) the exact formulas of (30) are recovered. Therefore the approximate formula (18) is well adapted to the multipath case.

3.1.3. Transmission with some modulation frequencies

(a) Let us now presume a dispersive medium characterized by a scattering function that exists only over the doppler axis, and is distributed on some discrete points of this axis:

$$\tilde{S}(\nu, \xi) = \sum_{i=1}^{N} q_i \delta(\nu - \nu_i) \delta(\xi).$$

This corresponds to a modulation of the signal $\tilde{f}(t)$ by a sum of monochromatic waves:

$$\tilde{s}(t) = \tilde{M}(t)\tilde{f}(t); \qquad \tilde{M}(t) = \sum_{j=1}^{N} \tilde{\alpha}_j e^{i2\pi\nu_j t},$$

where the $\tilde{\alpha}_j$ are always complex, independent, zero mean, gaussian variables, with respective variances $q_i = E\{|\tilde{\alpha}_i|^2\}$. We have encountered the case $N = 1$ in vertical submarine acoustical propagation.

The covariance of $\tilde{s}(t)$ is

$$\tilde{\Gamma}_{\tilde{s}}(t, u) = \tilde{f}(t)\tilde{\Gamma}_{\tilde{M}}(t, u)\tilde{f}^*(u),$$

with

$$\tilde{\Gamma}_{\tilde{M}}(t, u) = \sum_j q_j e^{i2\pi\nu_j(t-u)}. \tag{32}$$

The form (32) is still of the factorable form (24), in which:

$$A = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_N \end{bmatrix} \quad \text{and} \quad \tilde{\psi}(t) = \begin{bmatrix} \tilde{f}(t) \, e^{i2\pi\nu_1 t} \\ \vdots \\ \tilde{f}(t) \, e^{i2\pi\nu_N t} \end{bmatrix}.$$

The matrix $\boldsymbol{A}$ is diagonal and the vector $\tilde{\psi}(t)$ is composed of N doppler-delayed signals. The optimal receiver has really the same structure as that given above (formulas (29), (30) and Fig. (3)), by replacing the vector $\tilde{\psi}$ by the vector given by (44).

The C_{kj} coefficients are now:

$$C_{kj} = \int_0^T |\tilde{f}(t)|^2 \, e^{i2\pi(\nu_k - \nu_j)t} \, dt \triangleq \tilde{\chi}_{\tilde{f}}(0, \nu_j - \nu_k),$$

where $\tilde{\chi}_{\tilde{f}}(\tau, \nu)$ is the translation time-frequency ambiguity function of $\tilde{f}$:

$$\tilde{\chi}_f(\tau, \nu) = \int \tilde{f}(t) \tilde{f}^*(t - \tau) \, e^{-i2\pi\nu t} \, dt. \tag{33}$$

The coefficients C_{jk} correspond to the values of the doppler axis of the $\tilde{\chi}$ function (in the above paragraph, they were the values of the delay axis of $\tilde{\chi}$).

(a.1) If the minimum frequency range between the ν_j is greater than the inverse of T_0, the matrix $\boldsymbol{C}$ is diagonal, and the receiver is simplified as above.

(a.2) If there is only one doppler modulation $\nu_j = F$, in addition to the 'direct transmission' ($\nu_j = 0$), the situation is obviously parallel to the two-path case. If the second function $\tilde{f}(t - L)$ is replaced by $\tilde{f}(t) \, e^{i2\pi Ft}$ the results shown for performance in the two-path case are still valid.

(b) A similar situation is encountered if the scattering function is composed exclusively of Dirac functions whatever the position of these functions in the $\nu - \xi$ plane. Let:

$$\tilde{S}(\nu, \xi) = \sum_j \sum_k q_{jk} \delta(\nu - \nu_k) \delta(\xi - \xi_j). \tag{34}$$

The covariance of $\tilde{s}(t)$ is by substituting (33) into (4):

$$\tilde{\Gamma}_{\tilde{s}}(t, u) = \sum_j \sum_k \tilde{f}(t - \xi_j) \tilde{f}^*(u - \xi_j) \, e^{i2\pi\nu_k(t-u)} q_{jk} \triangleq \sum_{n=1}^N q_n \tilde{\psi}_n(t) \tilde{\psi}_n^*(u), \tag{35}$$

with:

$$\tilde{\psi}_n(t) = \tilde{f}(t - \xi_j) \, e^{i2\pi\nu_k t}, \qquad q_n = q_{jk}. \tag{36}$$

We define as many $\tilde{\psi}_n(t)$ functions as couples (ξ_j, ν_k) of $\tilde{S}(\nu, \xi)$. N is the total number of 'Dirac functions' in the (ν, ξ) plane.

The form (35) is still a particular case of (24), in which the matrix $\boldsymbol{A}$ is always diagonal, and the functions $\tilde{\psi}_n(t)$ are given by (36).

The matrix $\boldsymbol{C}$ is diagonal only if the position of Dirac functions in the (ν, ξ) plane are more separated than the time-frequency basis of the ambiguity function $\tilde{\chi}$ (that can be approximated by the inverse of the signal duration and bandwidth). Obviously this can only be an approximation because of the time-frequency duality.

Up to now we have seen different cases—which correspond to a ν, ξ quantized scattering function—where the optimal receiver can be obtained directly.

We now examine the opposite case, in which the scattering function is continuous over the (ν, ξ) plane. There may be different situations: if $\tilde{S}(\nu, \xi)$ is continuous with respect to ξ, in all cases its evolution $\tilde{S}(\xi)$ must be quantized (for example segmented). If $\tilde{S}(\nu, \xi)$ is continuous with respect to ν, we have seen above that the knowledge of a state variable model of $\tilde{s}(t)$ enables us to obtain the optimal receiver—which itself exhibits a state variable structure. We call this the indirect (state variable) realization of the optimal receiver, in opposition to the case given above.

3.2. Scattering function continuous over the (ν, ξ) plane

3.2.1. Rational scattering function

In the most general case, the realization of the optimal receiver is obtained by means of a state variable ξ-parametrized model. On the one hand, this parametrization is difficult. On the other hand, the identification of the state variable model itself is not completely solved. A few simple forms of $\tilde{S}(\nu, \xi)$ are known, which lead to an internal model. Kurth [11] has studied some simulations. As an example, a Lorentzian function—i.e. unimodal with respect to ν—such as [20]:

$$\tilde{S}(\nu) = \frac{\tilde{Q}(\xi)}{(2\pi\nu + K_i(\xi))^2 + K_r^2(\xi)},$$

leads to a first order state variable model in which:
   (i) the transition matrix, called $\tilde{F}(\xi)$ in the annex, becomes $K_r(\xi) - jK_i(\xi)$.
   (ii) $\tilde{Q}(\xi)$ corresponds to the mean power of the white noise exciting the state variable model.

Obviously, such a first order model cannot represent every kind of scattering function; but it allows many possibilities.

If the scattering function is multimodal with respect to ν, a second or more order model must be adopted [19].

3.2.2. General multipath case

We have studied the multipath case in particular (see paragraph 3.1.2), but we do not make the 'ideal' transmission hypothesis.

Accordingly, there may be a signal time modulation along each path, and this fast fading leads to a frequential spreading of the scattering function. Its theoretical form is:

$$\tilde{S}(\nu, \xi) = \sum_{i=1}^{N} \tilde{S}_{di}(\nu)\delta(\xi - \xi_i).$$

A state variable model can be chosen for each frequency spread path (by the same method as described in the Annex), provided that $\tilde{S}_{di}(\nu)$ is rational with respect to ν. When considering the complete transmission, all the state variable models can be put into a vectorial state variable model.

We give below an example of an experimental submarine scattering function, with an obvious multipath structure. In this case, we give as an illustration, the results obtained by keeping five major paths ξ_1, ξ_2, ξ_3, ξ_4, ξ_5.

Here, the state variable internal model corresponding to each path is a first order model for all the evolutions with respect to ν are quasi Lorentzian. Let $1/K_i$ be the time constant of the respective first order model. The quantities $q_i\bar{E}_r$ ($i = 1 \ldots 5$) are the mean powers transmitted over each path. It is possible to show [7] that the corresponding optimal signal estimate (main part of the optimal receiver) has the structure given in Fig. 7.

This optimal estimate possesses five 'branches'. Over each branch, the receiver is itself a first order system, and after that, there is a correlation with the expected form: $\tilde{f}(t - \xi_i)$. In this model, we have assumed:

$$\tilde{S}_{di}(\nu) = \frac{2K_i q_i}{K_i^2 + 4\pi^2\nu^2}. \tag{37}$$

By normalizing the quantities $q_i(\sum q_i = 1)$, the constants $-K_i = -K$ corresponding to Fig. 6 are about 0.25 Hz.
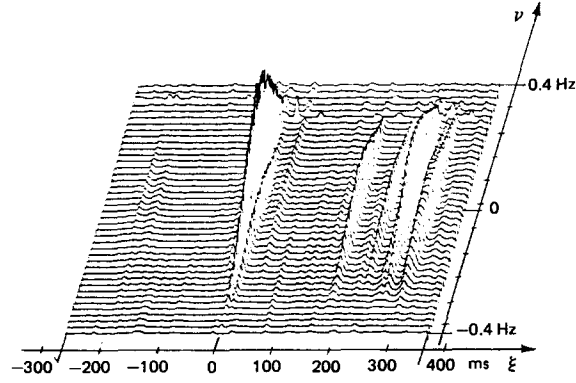
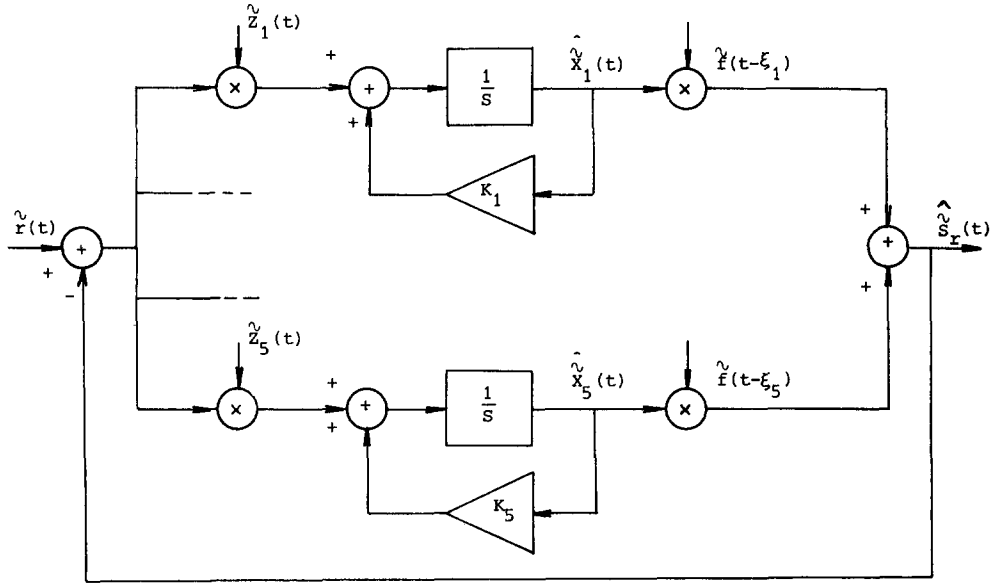Fig. 6. Example of an estimated scattering function.

Fig. 7. Optimal internal estimate of the signal in a non-ideal multipath transmission.

The matrix and vector equations of the optimal receiver are:

$$\dot{\hat{\tilde{X}}} = \tilde{F}\hat{\tilde{X}}(t) + \tilde{Z}(t)[\tilde{r}(t) - \tilde{C}^\dagger(t)\hat{\tilde{X}}(t)], \quad \hat{\tilde{X}}(0) = 0,$$

$$\hat{\tilde{s}}_r(t) = \tilde{C}^\dagger(t)\hat{\tilde{X}}(t),$$

where the matrices F and C are defined by:

$$\tilde{F} = K\mathbb{1}; \qquad \tilde{C}^\dagger(t) = E[\tilde{f}(t - \xi_1) \dots \tilde{f}(t - \xi_5)].$$

The Kalman gain matrix is:

$$\tilde{Z}(t) = \tilde{P}(t)\tilde{C}(t),$$

where $\tilde{P}(t)$ satisfies the Riccati differential equation:

$$\frac{\mathrm{d}\tilde{P}(t)}{\mathrm{d}t} = 2K\tilde{P}(t) + \tilde{Q} - \tilde{P}(t)\tilde{C}(t)\frac{1}{N_0}C^\dagger(t)P(t), \quad t \geq 0, \tag{38}$$

$$\tilde{P}(0) = -\frac{\tilde{Q}}{2K}, \quad \tilde{Q} = \begin{bmatrix} 2kq_1 & & \\ & \ddots & \\ & & 2kq_5 \end{bmatrix}.$$

Let us remark that the one or two frequency-spread paths cases has been studied more extensively [3]. In particular, the differential equation (38) has been solved numerically and the mean square time variant estimation error has been obtained. An interesting result is that the best receiver in the Bayes sense does not lead to a small estimation error for the estimate $\hat{s}_r(t)$. Let us note that another receiver has been studied, using a simpler quasi optimal receiver, which might exempt the Kalman gain calculus, and the resolution of (38), [23].

So, even when the scattering function exists in the entire plane (ν, ξ), the structure of the optimal receiver can be obtained. When it is discretized with respect to ξ (as in the example above), the receiver state variable internal model is simplified and it is reasonable to think of an effective realization (for example, see Fig. 7).

4. Performance and emission signal design

It is well known that in a dispersive transmission, the signal energy is not sufficient to determine the receiver performance. In order to know what signal $\tilde{f}(t)$ is best in a given dispersive transmission, it would be necessary to know the output statistic *pdf*; that is, the error probability in binary communication, and from that derive the best form of the emitted signal. In some particular cases (see ideal multipath case) we have exact expressions. But in the most general case, these output probabilities are only approached, as we have seen above, or bounded [19].

4.1. Some general new results

The formulas commonly used have been recalled in paragraph 2.3.1. In particular, the quantity $\mu(1/2)$ given by (15) is an efficiency measure of a communication system.

Kennedy has shown that an optimum value of $\mu(1/2)$ can be obtained for different combinations of signals $\tilde{f}(t)$ and spread scattering functions if it is possible to design the signal $\tilde{s}(t)$ so that all its eigenvalues are identical and equal to $(\bar{E}_r/N_0) \cdot (1/3.07)$ (implicit diversity).

It is not evident that this optimum configuration can be reached. Moreover, we have no idea about the means for reaching it.

The approximation formulas (18–23) permit a direct connection to be made between the error probability and the emitted signal. In fact, the only quantities entering into the error probability expression are the output first and second order moments. The first moment is (cf. (20)) the received SNR, which is usually given. The second moment is $\mathrm{Var}\{l/H_1\}$ (given by (24)) which may be expressed in terms of the normalized medium scattering function $\tilde{S}_n(\nu, \xi) = \tilde{S}(\nu, \xi)/M$ (M given by (6)), and the normalized ambiguity function:

$$\tilde{\psi}_{\tilde{f}}(\tau, \nu) \triangleq \frac{1}{E}\tilde{\chi}_{\tilde{f}}(\tau, \nu), \quad \tilde{\chi}_{\tilde{f}} \text{ given by (33).} \tag{39}$$

With (39) and (4), (21) is written as below:

$$\text{Var}\{l/H_1\} = \left(\frac{\bar{E}_r}{N_0}\right)^2 \int_{R^4} \tilde{S}_n(f, \xi)\tilde{S}_n(\nu, \tau)|\tilde{\psi}_{\tilde{f}}(\xi - \tau, f - \nu)|^2 \, d\tau \, d\nu \, d\xi \, df. \tag{40}$$

Thus we have a direct relationship between the receiver performance and the signal ambiguity function. Performance variation versus the quantity:

$$b = \text{Var}\{l|H_1\}\frac{N_0}{\bar{E}_r},$$

has been studied [15].

It has been shown that, for a given received SNR, a value of b exists that minimizes the error probability. The parameter b—which is a SNR per diversity path—for high received SNR, approaches 3.5 for detection problems (eq. 7) and 3 for binary communication problems (eq. 8). *So when knowing the optimal value of b, one must point out the optimal value of the ambiguity function given by (40), and consequently the optimal signals.*

For some values of SNR the optimal value of b cannot be reached: we may see this by studying in greater detail the variance defined by equation (40).

If the scattering function can be reduced to $\delta(\xi)\delta(\nu)$ (this is the Rayleigh model), so that

$$\text{Var}\{l|H_1\} = (\bar{E}_r/N_0)^2, \tag{41}$$

the signal form is no longer of relevance to performance.

On the contrary, if the signal is a large time and bandwidth signal, so that its ambiguity function can be approached by a double Dirac function then:

$$\text{Var}\{l|H_1\} = \left(\frac{\bar{E}_r}{N_0}\right)^2 \iint \tilde{S}_n^2(f, \xi) \, df \, d\xi. \tag{42}$$

In fact, the variance is always included between these two bounds; let:

$$V = \frac{\text{Var}\{l|H_1\}}{(\bar{E}_r/N_0)^2} = b/\bar{E}_r/N_0$$

be the double integral in (40). V can be interpreted as the two-dimensional scalar product of $\tilde{S}_n(f, \xi)$ with a quantity $\tilde{A}(f, \xi)$ that is defined to be the two-dimensional convolution of $\tilde{S}_n$ and $|\tilde{\psi}_{\tilde{f}}|^2$:

$$\tilde{A}(f, \xi) \triangleq \tilde{S}_n \underset{2\,\text{dim}}{*} |\tilde{\psi}_{\tilde{f}}|^2, \tag{43}$$

$$V = \underset{2\,\text{dim}}{\langle \tilde{S}_n, \tilde{A} \rangle}.$$

The quantities entering into this convolution are real and positive; so the scalar product of $\tilde{S}_n$ and $\tilde{A}$ is greater than that between $\tilde{S}_n$ and $\tilde{S}_n$:

$$\underset{2\,\text{dim}}{\langle \tilde{S}_n, \tilde{S}_n \rangle} \leq V.$$

So:

$$\left(\frac{\bar{E}_r}{N_0}\right)^2 \cdot \iint \tilde{S}_n^2(f, \xi) \, df \, d\xi \leq \text{Var}\{l|H_1\}. \tag{44}$$

On the other hand, the two-dimensional Schwarz inequality leads to:

$$\langle \tilde{S}_n, \tilde{A} \rangle \leqslant \sqrt{\langle \tilde{S}_n, \tilde{S}_n \rangle \langle \tilde{A}, \tilde{A} \rangle},$$

(The scalar products are all two-dimensional ones).

Moreover, $\tilde{S}_n(\nu, \xi)$ is normalized; so $\langle \tilde{S}_n, \tilde{S}_n \rangle \leqslant 1$.

In the same way, let us show that $\tilde{A}$ is also normalized. Assume the following bidimensional Fourier transforms (2 dim F.T.):

$$\tilde{a}(t, \nu) \underset{2\,\mathrm{dim}}{\rightleftharpoons} \tilde{A}(f, \xi); \qquad \tilde{\Gamma}(t, \nu) \underset{2\,\mathrm{dim}}{\rightleftharpoons} \tilde{S}_n(f, \xi)$$

and $|\tilde{\psi}_{\tilde{f}}|^2$ is its own 2 dim F.T.

$$\iint \tilde{A}(f, \xi)\, \mathrm{d}f\, \mathrm{d}\xi = \tilde{a}(0, 0) = \tilde{\Gamma}(0, 0) \cdot |\tilde{\psi}_{\tilde{f}}(0, 0)|^2.$$

Or:

$$\tilde{\Gamma}(0, 0) = \iint \tilde{S}_n(\nu, \xi)\, \mathrm{d}\nu\, \mathrm{d}\xi = 1 \text{ and } |\tilde{\psi}_{\tilde{f}}(0, 0)|^2 = 1.$$

So: $\tilde{a}(0, 0) = 1$ and by the same reasoning as given above:

$$\langle \tilde{A}, \tilde{A} \rangle = \iint \tilde{A}^2(f, \xi)\, \mathrm{d}f\, \mathrm{d}\xi \leqslant 1.$$

So:

$$\langle \tilde{S}_n, \tilde{A} \rangle \leqslant 1 \text{ and } \mathrm{Var}\{l|H_1\} \leqslant \left(\frac{E_r}{N_0}\right)^2. \tag{45}$$

These two bounds (44) and (45) are illustrated in the following scheme where we have given $\mathrm{Var}\{l|H_1\}$ in terms of S.N.R. The nearly linear curve is the optimal value needed to minimize the error probability and the textured zone is the possible zone for $\mathrm{Var}\{l|H_2\}$. It is limited by the two above bounds.

We see therefore that it is not always possible to choose an optimal value for the variance. The inferior bound depends on the medium itself. For example we have plotted it for the case of two identical paths [see example below (Section 4.2a)]: in this case we see that when the S.N.R. is below 6, it is possible to choose the optimal value; when the S.N.R. is larger, it is only possible to reach the minimal variance. The optimal variance would be reached if there were more propagation paths in order to 'distribute' $\bar{E}_r/N_0$ in the best way: it is an 'implicit diversity' system [24].

4.2. Examples

(a) When the medium is only time dispersive, the scattering function becomes $\tilde{S}(\nu, \zeta) = \tilde{Q}(\xi)\delta(\nu)$ and the parameter b is now:

$$b = \frac{\bar{E}_r}{N_0} \iint \tilde{Q}(\zeta)\tilde{Q}(\tau)|\tilde{\psi}_{\tilde{f}}(\zeta - \tau, 0)|^2\, \mathrm{d}\xi\, \mathrm{d}\tau,$$

where $\tilde{\psi}_{\tilde{f}}$ is always a normalized ambiguity function of $\tilde{f}(t)$. In this case the parameter of interest in the performance is only the signal correlation function $\tilde{\psi}_f(\tau, 0)$, i.e. *the main parameter will be the effective signal bandwidth*. The choice of *frequency or phase modulation* is important. We have seen above the
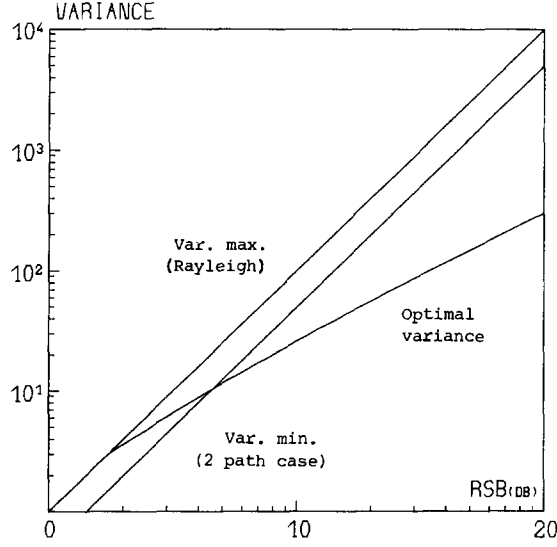
Fig. 8. Choice of the variance.

particular case of ideal two-path case: in this case the only parameter which appears in the error probability calculation is the correlation coefficient p for the delay between the two paths. *The most favourable case is always* $p = 0$.

(b) When the medium is only frequency dispersive, the scattering function becomes $\tilde{S}(\nu, \xi) = \tilde{S}(\nu)\delta(\xi)$ and the parameter b is:

$$b = \frac{\bar{E}_r}{N_0} \int \int \tilde{S}(\mu)\tilde{S}(\nu)|\tilde{\psi}_f(0, \mu - \nu)|^2 \, d\nu \, d\mu, \tag{46}$$

where $\tilde{\psi}_f(0, \nu)$ is the doppler resolution function of $\tilde{f}(t)$. In fact:

$$\tilde{\psi}_f(0, \nu) \;\rightleftharpoons\; |\tilde{f}(t)|^2. \tag{47}$$

So in this case, only the *amplitude modulation* can influence the performance. *Here the important parameter is the effective signal duration.*

Different new numerical simulations have been carried out in the binary symmetric communication case [2]. The scattering function is assumed to have a Lorentzian form (given by the expression (37) above) and normalized: so, $q_i = q = 1$. In this work, the performance has not been evaluated by calculating the parameter b but by calculating $\mu(1/2)$.

The quantity $\mu(1/2)$ has been studied for the four forms of amplitude modulation given above in the first column, in order to compare these different forms.

It has been established that, by considering the 'effective duration', T_u, defined by Kennedy [10] as:

$$T_u = E^2 \cdot \left[\int_0^{T_0} |\tilde{f}(t)|^4 \, dt \right]^{-1}. \tag{49}$$

The forms given above are quasi-identical from the view of performance with respect to $\mu(1/2)$. For the first form, $T_u = T_0$. Figure 10, given below, was obtained numerically, and the performance $\mu(1/2)$ versus the mean received SNR for different values of the product KT_u (K is the channel frequential spreading given by the scattering function).
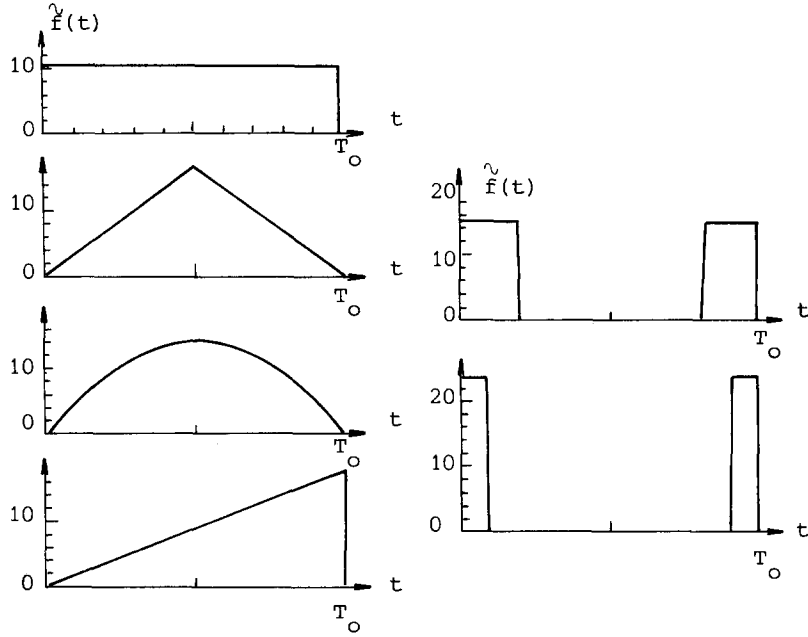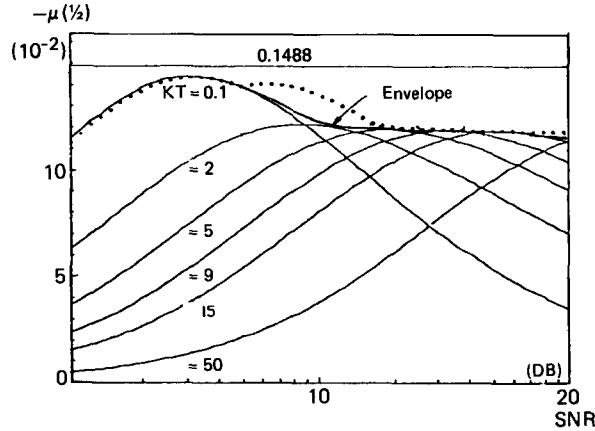
Fig. 9. Different amplitude modulations.

Fig. 10. Performance versus S.N.R. for the first form of $\tilde{f}(t)$.

A maximum value of $\mu(1/2)$ exists for each curve, occurring for a SNR that verifies the relation:

$$\frac{\bar{E}_r/N_0}{N} \sim 3.4, \quad \text{with } N = 1 + KT_u. \tag{67}$$

N is the diversity number in the Kennedy sense [10].

We have confirmed here the effectiveness of the parameter T_u for evaluating new signals, and we have recovered the result given above by means of the new approximate formulas of the error probability (formula (22)).

So, for a given received SNR and a given dispersive channel (K) one can choose the optimal T_u; and by optimizing this choice for each configuration, the performance will be characterized by the envelope curve (Fig. 10 above), which is not far from the optimum (given by 0.1488).

Another fifth signal form has been studied, which is more readily adapted to temporal diversity: the coded signals no. 1 and 2 are given in the second column of Fig. 9. Similar results are obtained with respect to the quantity T_u. However, we have observed that with the second code, the evolution of $\mu(1/2)$ versus SNR remains for a longer interval very near to the optimum $\mu(1/2)$. It corresponds to the dotted lines of Fig. 10. In this sense, this second type of signal is preferable. It means that the two impulses of the signal $\tilde{f}(t)$ are distant enough for decorrelation versus the channel correlation time $1/K$.

This is analogous to a temporal diversity. Nevertheless, let us remark that the longer the duration of $\tilde{f}(t)$ the more reduced is the information rate.

(c) *Time and frequency dispersive channel.* We have studied the case of two L-delayed paths, and frequency dispersive (in the sense of the case given above):

$$\tilde{S}(\nu, \zeta) = \tilde{S}_1(\nu)\delta(\zeta) + \tilde{S}_2(\nu)\delta(\xi - L).$$

The parameter b becomes the summation of four terms in which the doppler resolution of the signal always occurs, but in addition the value of the ambiguity function for a delay L is also of interest.

$$b = \frac{\bar{E}_r}{N_0}\left[\iint \tilde{S}_1(f)\tilde{S}_1(\nu)|\tilde{\psi}_{\tilde{f}}(0, f - \nu)|^2 \, df \, d\nu + \iint \tilde{S}_2(f)\tilde{S}_2(\nu)|\tilde{\psi}_{\tilde{f}}(0, f - \nu)|^2 \, df \, d\nu \right.$$

$$\left. + 2 \iint \tilde{S}_2(f)\tilde{S}_1(\nu)|\tilde{\psi}_{\tilde{f}}(L, f - \nu)|^2 \, df \, d\nu \right].$$

When the frequency spreading is assumed to be of the same type as above (with parameters K_1 and K_2 respectively), we have observed [2] the influence of both kinds of parameters K_iT_u and L/T. It appears that the conclusions obtained in the separate cases—only time dispersive and only frequency dispersive—are valid and can be gathered together.

5. Conclusion

In this paper, we have developed techniques for elaborating an optimal receiver at the output of a dispersive channel that is characterized by a scattering function estimated previously.

In most cases, it is possible to *elaborate the optimal receiver*: in fact when the scattering function can be considered to be discrete in the delay-doppler plane, the optimal receiver is given explicitly in paragraph 3.1. The most difficult case occurs when the scattering function is continuous with respect to delay time: the evolution along this axis must be quantized. When the scattering function is continuous with respect to doppler shift, the optimal receiver analysis must be carried out by means of an internal model (ex. paragraph 3.2). Nevertheless some new results permit us to assume that good results can also be obtained by using a simplified Kalman filter [23].

These results show the importance of the quality of the estimation of the medium scattering function; particularly of the plus or minus quantized character of this function.

But in many cases it is *still very difficult to calculate the optimal receiver performance exactly* and so to compare it with a suboptimal receiver. We have shown here a new way to consider the calculation of performance and of signal design (paragraph 4).

The major new results are in paragraph 4: first the general evolution of the variance of the statistics and the domain where it will be possible to adapt signal ambiguity exactly to the scattering function (Fig. 8). In addition, we have new results for the only frequency dispersive case, where we have compared

several amplitude modulation forms and we have exhibited the importance of the parameter T_u and its connexion to temporal diversity. These connections are also exhibited in [24] in another way.

Annex

State variable model of $\tilde{s}(t)$ and of the optimal estimate $\hat{\tilde{s}}_r(t)$ [17].

(a) The general structure of the state variable model of $\tilde{H}(t, \xi)$ and $\tilde{s}(t)$ is given in Fig. A1 in which $\tilde{U}(t, \xi)$ is the noise incoming to the state variable model.
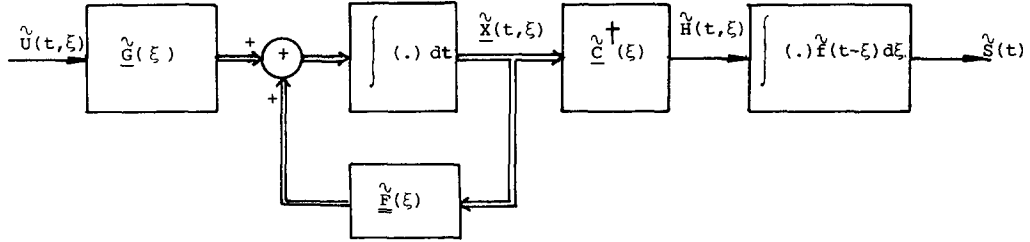

Fig. A1. State variable model of H and s.

The whole model (input noise, and matrix systems) is parametrized versus ξ, in order to obtain the random bivariable function $\tilde{H}(t, \xi)$; the state model itself is related to the variable t. But let us recall the hypothesis WSSUS enables to simplify the model because:

(i) the matrices $\tilde{F}, \tilde{G}, \tilde{C}$ are independent of t;

(ii) the input noise is stationary versus t and ξ decorrelated.

The corresponding equations of Fig. A1 are:

$$\frac{\partial}{\partial t} \tilde{X}(t, \xi) = \tilde{F}(\xi)\tilde{X}(t, \xi) + \tilde{G}(\xi)\tilde{U}(t, \xi), \quad t > 0. \tag{A2}$$

The noise $\tilde{U}(t, \xi)$ is zero-mean, with the covariance:

$$E\{\tilde{U}(t, \xi)\tilde{U}(t', \zeta)\} = \tilde{Q}(\xi)\delta(t - t')\delta(\xi - \zeta), \tag{A3}$$

$$\tilde{H}(t, \xi) = \tilde{C}^\dagger(\xi)X(t, \xi).$$

$\tilde{F}(\xi)$ is the state transition matrix, $\tilde{G}(\xi)$ the input vector, $\tilde{C}(\xi)$ the observation vector.

The system order, as well as the matrices $\tilde{F}, \tilde{G}, \tilde{C}$, is defined from the form of the scattering function. The state vector initial covariance is:

$$E\{\tilde{X}(0, \xi)\tilde{X}^\dagger(0, \zeta)\} = \tilde{P}_0(\xi)\delta(\xi - \zeta), \tag{A4}$$

in which $\tilde{P}_0(\xi)$ is the solution of the matricial differential equation:

$$\tilde{F}(\xi)\tilde{P}_0^\dagger(\xi) + \tilde{P}_0(\xi)\tilde{F}^\dagger(\xi) + \tilde{G}(\xi)\tilde{Q}(\xi)\tilde{G}^\dagger(\xi) = 0. \tag{A5}$$

So we have the state model corresponding to the signal $\tilde{s}(t)$ given by:

$$\tilde{s}(t) = \int \tilde{f}(t - \xi)\tilde{C}^\dagger(\xi)\tilde{X}(t, \xi) \, d\xi. \tag{A6}$$

(b) The structure of the state variable model optimal receiver is deduced directly from the $s(t)$ model. The estimating structure with the internal model is given below, with the corresponding equations [17]. In fact $\tilde{H}(t, \xi)$ is first estimated, then $\tilde{s}(t)$ is estimated in order to obtain $\hat{\tilde{s}}_r(t)$. The matrices $\tilde{F}, \tilde{G}, \tilde{C}$ are obviously the same as in the above model of $\tilde{s}(t)$.

the estimate equations are:

$$\frac{\partial}{\partial t} \hat{\tilde{X}}(t, \xi) = \tilde{F}(\xi) \hat{\tilde{X}}(t, \xi) + Z(t, \xi)[\tilde{r}(t) - \hat{\tilde{s}}_r(t)], \quad t > 0, \tag{A7}$$

$$\hat{\tilde{X}}(0, \xi) = 0,$$

$$\hat{\tilde{s}}_r(t) = \int \tilde{f}(t - \xi) \tilde{C}^\dagger(\xi) \hat{\tilde{X}}(t, \xi) \, d\xi. \tag{A8}$$

The matrix of Kalman gain $Z(t, \xi)$ is given by:

$$\tilde{Z}(t, \xi) = \frac{1}{N_0} \int \tilde{P}(t, \xi, \tau) \tilde{C}(\tau) \tilde{f}^*(t - \tau) \, d\tau. \tag{A9}$$

Where $\tilde{P}(t, \xi, \tau)$ satisfies the Riccati matricial equations:

$$\frac{\partial \tilde{P}(t, \xi, \tau)}{\partial t} = \tilde{F}(\xi) \tilde{P}(t, \xi, \tau) + \tilde{P}(t, \xi, \tau) \tilde{F}^\dagger(\tau) + \tilde{G}(\xi) \tilde{Q}(\xi) \tilde{G}^\dagger(\xi)$$

$$- \frac{1}{N_0} \left[\int \tilde{P}(t, \xi, \sigma) \tilde{C}(\sigma) \tilde{f}^*(t - \sigma) \, d\sigma \right] \left[\int \tilde{f}(t - \sigma) \tilde{C}^\dagger(\sigma) \tilde{P}(t, \sigma, \tau) \, d\sigma \right],$$

with $\tilde{P}(0, \xi, \tau) \equiv \tilde{P}_0(\xi) \delta(\xi - \tau)$, $\tilde{P}_0$ is given by (A4).

So we have obtained the optimal realizable structure of the estimate signal $\hat{\tilde{s}}_r(t)$ and the optimal bayesian receiver is obtained by reporting the Fig. A2 in the dotted 'box' of Fig. 2.

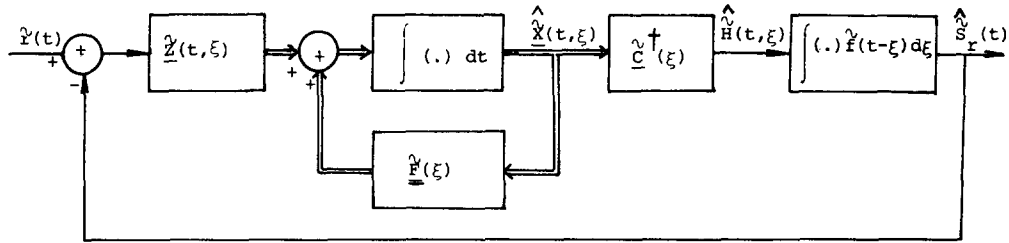Fig. A2. State variable model of the optimal receiver.

This optimal system is obviously very difficult to realize, the main difficulty being the ξ parametrization. Let us recall that Kurth has proposed a theoretical method of development of the variance $P(t, \xi, \tau)$ into orthogonal functions, in order to try to realize this structure.

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