On the Distribution of Positive-Definite Gaussian Quadratic Forms

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Abstract —Quadratic signal processing is used in detection and estimation of random signals. To describe the performance of quadratic signal processing, the probability distribution of the output of the processor is needed. Only positive-definite Gaussian quadratic forms are considered. The quadratic form is diagonalized in terms of independent Gaussian variables and its mean, moment-generating function, and cumulants are computed; conditions are given for the quadratic form to be χ^2 distributed and distributed like a sum of independent random variables having a Gamma distribution. A new method is proposed to approximate its probability distribution using an expansion in Laguerre polynomials for the central case and in generalized χ^2 distributions in the noncentral case. The series coefficients and bounds on truncation error are evaluated. Some applications in average power and power spectrum estimation and in detection illustrate our method.

I. Introduction

Quadratic processing of signals is used in many problems of statistical communication, detection, and estimation. The reason is that the logarithm of the likelihood ratio of two equivalent Gaussian measures is a quadratic functional of the process, if the covariance operators differ [25]. The case of detection or reception of Gaussian signals in Gaussian noise is studied by Kadota [17], [18]. The quadratic functional is positive-definite and the properties and methods given here can be applied. Also, to estimate certain second order characteristics of Gaussian processes, such as power or power spectrum, one frequently uses quadratic functionals as estimators.

There is a vast literature on the probability distribution of a quadratic form especially in the finite-dimensional case. The more relevant references and results are given by Johnson and Kotz [16]. In this paper we propose a method of approximation by expanding the distribution in Laguerre polynomials in the central case and in generalized noncentral χ^2 distributions in the noncentral case. This method can be considered as an extension of known techniques from finite to infinite quadratic forms.

We shall first define the quadratic form we will consider. It is well-known that we can consider Gaussian signals as elements of a Hilbert space. This approach gives a geometric interpretation that is interesting and advantageous in problems of classification and estimation. Except in some degenerate cases, one needs an infinite-dimensional Hilbert space to represent a continuous-time random process. The connection between Gaussian signals and Gaussian measures in a linear space is explained by Rajput and Cambanis [24].

Let $(\Omega, \mathscr{A}, P)$ be a probability space, H be a complex or real separable Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and $\mathscr{B}$ be the σ-algebra of all Borel subsets of H. A random element X with values in H induces a probability measure ν on $(H, \mathscr{B})$ by $\nu(B) = P\{X \in B\}$, $B \in \mathscr{B}$; ν is the distribution of X [5]. This measure is Gaussian if, for every $y \in H$, the random variable $\langle X, y\rangle$ is Gaussian. In this paper we consider either real or complex random variables. The mean element $m \in H$ of X satisfies, for all $y \in H$,

$$\langle m, y\rangle = \boldsymbol{E}\{\langle X, y\rangle\} = \int_H \langle x, y\rangle \, d\nu(x)$$

and the covariance operator K of X satisfies, for all $u \in H$ and $v \in H$,

$$\langle Ku, v\rangle = \operatorname{cov}\{\langle X, u\rangle, \langle X, v\rangle\}$$
$$= \int_H \langle x - m, u\rangle^* \langle x - m, v\rangle \, d\nu(x).$$

K is a nuclear, positive-definite, self-adjoint linear operator in H [22].

We define a quadratic form (or functional) by the inner product

$$Q = \langle TX, X\rangle \tag{1}$$

where X is a Gaussian element in H and T is a nonnegative-definite self-adjoint bounded linear operator. We want to determine the probability distribution of the nonnegative random variable Q. We approximate the probability density function of Q by expanding it in a series. In the central case we choose the gamma distribution as the basis, or first approximation, of the expansion and we use an expansion in a convergent series of Laguerre polynomials. To evaluate series coefficients we use the traces of the operator KT that utilize its eigenvalues in a concise way. We give conditions for convergence, algorithms for calculations, and error bounds when approximations by truncation are considered. In the noncentral case we use an expansion in a series of functions derived from the

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noncentral χ^2 distribution. These methods and derivations are the object of Section III. In Section II we give the necessary theoretical background concerning quadratic forms in Gaussian random elements: a diagonalization in independent Gaussian variables, the moment-generating function, and the cumulants of the quadratic form. In particular, we give a new necessary and sufficient condition for a Gaussian quadratic form to be χ^2 distributed, central or not, and to be distributed like a sum of independent random variables having a Gamma distribution. Finally in Section IV we give some applications to detection and estimation problems that illustrate the proposed method, indicate its efficiency, and permit comparison with other methods.

II. Properties of Quadratic Functionals

We shall study the properties of the quadratic functional (1) under the assumptions of the introduction. We can assume without loss of generality that the covariance operator K is strictly positive-definite, so the range of its square root is dense in H (i.e., $\overline{\mathscr{R}(K^{1/2})}=H$). We demonstrate that the quadratic functional Q can be diagonalized, using the eigenvectors and the eigenvalues of the compact operator $K^{1/2}TK^{1/2}$. Ibragimov [15], using a different method, proved that Q has the same distribution as the sum given below in (2) (convergence in distribution). We also give expressions for the mean, the moment-generating function and the cumulants of Q, using the mean of X and the operator KT.

Proposition 1: The quadratic functional Q can be diagonalized

$$Q=\sum_i \lambda_i |Z_i|^2, \tag{2}$$

where $\{\lambda_i\}$ and $\{e_i\}$ are the eigenvalues and eigenvectors of the operator $K^{1/2}TK^{1/2}$, and the Gaussian variables $Z_i=\langle X, K^{-1/2}e_i\rangle$ are independent with mean $\langle m, K^{-1/2}e_i\rangle$ and variance 1. The mean of Q is

$$E\{Q\}=\langle Tm, m\rangle+\operatorname{tr}[KT], \tag{3}$$

its moment-generating function is

$$\Theta(s)=E\{\exp(-sQ)\}$$
$$=\frac{\exp\left(-s\left\langle T(I+sKT)^{-1}m, m\right\rangle\right)}{\det(I+sKT)} \tag{4}$$

for all s with $\lambda_{\max}\operatorname{Re}[s]>-1$, where $\lambda_{\max}$ is the largest eigenvalue of $K^{1/2}TK^{1/2}$, and its cumulants are

$$\kappa_n=(n-1)!\operatorname{tr}\left[(KT)^n\right]+n!\left\langle T(KT)^{n-1}m, m\right\rangle. \tag{5}$$

Proof: Since T is bounded, nonnegative-definite, and self-adjoint, and for all $u\in H$, $\langle K^{1/2}TK^{1/2}u, u\rangle \le \|T\|\langle Ku, u\rangle$, using a theorem by Douglas [7], we deduce that there exists a unique bounded operator U, such that

$$(K^{1/2}TK^{1/2})^{1/2}=K^{1/2}U \tag{6}$$

and the null space $\mathscr{N}(U)=\mathscr{N}((K^{1/2}TK^{1/2})^{1/2})$. Then we can write

$$\sum_i \lambda_i|\langle X, K^{-1/2}e_i\rangle|^2=\sum_i\left|\left\langle X, K^{-1/2}(K^{1/2}TK^{1/2})^{1/2}e_i\right\rangle\right|^2$$
$$=\sum_i|\langle X, Ue_i\rangle|^2$$
$$=\sum_i|\langle U^*X, e_i\rangle|^2=\|U^*X\|^2$$
$$=\langle TX, X\rangle,$$

since by (6), $K^{1/2}TK^{1/2}=K^{1/2}UU^*K^{1/2}$, which implies that $T=UU^*$, and since the set of the vectors $\{e_i\}$ is a complete orthonormal basis in $\mathscr{R}(K^{1/2}TK^{1/2})$. When X is complex, the complex Gaussian variables $Z_i=\langle X, K^{-1/2}e_i\rangle$ have mean $\mu_i=\langle m, K^{-1/2}e_i\rangle$ and covariance $E\{(Z_i-\mu_i)(Z_j-\mu_j)^*\}=\langle KK^{-1/2}e_i, K^{-1/2}e_j\rangle=\langle e_i, e_j\rangle=\delta_{ij}$. Hence they are independent. Using (2) and the method used for the diagonalization of Q, we obtain

$$E\{Q\}=\sum_i\lambda_i\left(|\mu_i|^2+1\right)=\langle Tm, m\rangle+\operatorname{tr}[KT].$$

The moment-generating function is

$$\Theta(s)=E\{\exp(-sQ)\}$$
$$=\left[\prod_i(1+s\lambda_i)\right]^{-1}\exp\left(-\sum_i s\lambda_i|\mu_i|^2/(1+s\lambda_i)\right) \tag{7}$$

for every $s\in\mathscr{C}$ such that $\lambda_{\max}\operatorname{Re}[s]>-1$. The operator KT has the same eigenvalues as the operator $K^{1/2}TK^{1/2}$, and the operators $(I+sKT)^{-1}$ and $(I+sK^{1/2}TK^{1/2})^{-1}$ exist for s as above. Let us consider the exponent in the expression of the moment-generating function (7). When $m\in\mathscr{R}(K^{1/2})$, we can write

$$\sum_i\lambda_i|\mu_i|^2/(1+s\lambda_i)$$
$$=\sum_i\langle m, K^{-1/2}K^{1/2}TK^{1/2}e_i\rangle\cdot\langle K^{-1/2}(I+sK^{1/2}TK^{1/2})^{-1}e_i, m\rangle$$
$$=\sum_i\langle K^{1/2}Tm, e_i\rangle\cdot\left\langle e_i,(I+s^*K^{1/2}TK^{1/2})^{-1}K^{-1/2}m\right\rangle$$
$$=\left\langle K^{1/2}Tm,(I+s^*K^{1/2}TK^{1/2})^{-1}K^{-1/2}m\right\rangle,$$

since $m\in\mathscr{R}(K^{1/2})$, $K^{1/2}Tm\in\mathscr{R}(K^{1/2}TK^{1/2})$, and therefore

$$\sum_i\langle K^{1/2}Tm, e_i\rangle\langle e_i, u\rangle=\langle K^{1/2}Tm, u\rangle$$

for all u in H. It is easily shown that

$$(I+sK^{1/2}TK^{1/2})^{-1}K^{1/2}TK^{1/2}$$
$$=K^{1/2}TK^{1/2}(I+sK^{1/2}TK^{1/2})^{-1}$$
$$=K^{1/2}T(I+sKT)^{-1}K^{1/2}.$$

Using also the fact that by definition

$$\prod_i (1+s\lambda_i) = \det(I+sKT)$$

(where $\det(\cdot)$ is the Fredholm determinant), we obtain the expression (4). Thus we have demonstrated (4) for $m \in \mathscr{R}(K^{1/2})$. Since $T(I+sKT)^{-1}$ is bounded and $\mathscr{R}(K^{1/2})$ is dense in H, this holds by continuity for all m. The cumulant-generating function is obtained using the following relation $\ln(\det(I+V)) = \mathrm{tr}[\ln(I+V)]$ for each nuclear operator V [27]. The cumulants are obtained by expanding in series the operators $(I+sKT)^{-1}$ and $\ln(I+sKT)$ for $\lambda_{\max}|s|<1$. QED.

Remark 1: If the Hilbert space H is real, we obtain the following expressions, with $\lambda_{\max}\mathrm{Re}[s] > -1/2$:

$$\Theta(s) = [\det(I+2sKT)]^{-1/2} \cdot \exp\left(-s\left\langle T(I+2sKT)^{-1}m, m\right\rangle\right). \tag{8}$$

$$\kappa_n = 2^{n-1}\left[(n-1)!\,\mathrm{tr}[(KT)^n] + n!\left\langle T(KT)^{n-1}m, m\right\rangle\right]. \tag{9}$$

In what follows we give a necessary and sufficient condition for Q to have a χ^2 distribution. First we consider the central case. Khatri [19] gives necessary and sufficient conditions for Q to have a χ^2 distribution in this case. The following proposition is equivalent, but it is simpler and more useful.

Proposition 2: When X is a zero-mean complex (real) Gaussian vector, then Q is distributed like $b\chi^2(k)$ if and only if

$$\mathrm{tr}\left[(KT)^3\right]\mathrm{tr}[KT] = \left(\mathrm{tr}\left[(KT)^2\right]\right)^2, \tag{10}$$

and in this case the number of degrees of freedom is

$$k = 2(\mathrm{tr}[KT])^2 \big/ \mathrm{tr}\left[(KT)^2\right] \tag{11}$$

and the variance per degree of freedom is

$$b = \mathrm{tr}\left[(KT)^2\right] \big/ 2\,\mathrm{tr}[KT] \tag{12}$$

(and in the real case $k/2$ and $2b$, respectively).

A demonstration of this proposition follows immediately from the following lemma.

Lemma: With $q_n = \mathrm{tr}[(KT)^n] = \Sigma_i \lambda_i^n$, we have for $n \geq 3$,

$$q_n/q_{n-1} \geq q_{n-1}/q_{n-2} \tag{13}$$

and

$$\lim_{n\to\infty} q_n/q_{n-1} = \lambda_{\max}. \tag{14}$$

If $q_3/q_2 = q_2/q_1$, then all eigenvalues are equal to λ and $q_n/q_{n-1} = \lambda$ for all n.

Proof: It is sufficient to prove that

$$\sum_i \lambda_i^n \sum_i \lambda_i^{n-2} \geq \left(\sum_i \lambda_i^{n-1}\right)^2,$$

which follows from Schwartz's inequality

$$\sum_i (\lambda_i^{n/2})^2 \sum_i (\lambda_i^{(n-2)/2})^2 \geq \left(\sum_i \lambda_i (n+n-2)/2\right)^2$$

with equality, iff $\lambda_i^{n/2} = c\lambda_i^{(n-2)/2}$ for every i. The equality is true, iff all nonzero eigenvalues are equal. Since $\Sigma_i \lambda_i < \infty$, only a finite number of eigenvalues can be nonzero. We have

$$q_n/q_{n-1} = \sum_i \lambda_i^n \Big/ \sum_i \lambda_i^{n-1} \to \lambda_{\max}$$

as $n \to \infty$. To obtain the equality it is sufficient to have two terms equal. QED.

In the noncentral case a similar argument may be used to prove that the same conditions are necessary and sufficient for Q to have a noncentral χ^2 distribution. The noncentrality coefficient is then

$$c = \left\{2\,\mathrm{tr}[KT]/\mathrm{tr}\left[(KT)^2\right]\right\}\langle Tm, m\rangle$$

and c/2 in the real case.

We shall prove now a more general proposition, giving a necessary and sufficient condition for Q to be distributed as a sum of independent random variables having Gamma distribution.

Proposition 3: The quadratic functional Q of (1) is distributed like the sum $\Sigma_{i=1}^n b_i Y_i$, where Y_i are independent random variables having a standard Gamma distribution $\Gamma(a_i)$, if and only if

$$\begin{vmatrix} q_1 & q_2 & \cdots & q_{n+1} \\ q_2 & q_3 & \cdots & q_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n+1} & q_{n+2} & \cdots & q_{2n+1} \end{vmatrix} = 0. \tag{15}$$

Proof: Let us suppose that there exist $b_i \neq 0$ $(i = 1,\cdots,n)$ such that $Q = \Sigma_{i=1}^n b_i Y_i$ (in law). Then we can write, for every $k \geq 1$, $\Sigma_{i=1}^n (a_i+1)b_i^k = q_k$. Let us consider the equations from $k=1$ to $k=n+1$, with a_i+1 as unknown variables. Then we obtain

$$\begin{vmatrix} b_1 & b_2 & \cdots & b_n & q_1 \\ b_1^2 & b_2^2 & \cdots & b_n^2 & q_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_1^{n+1} & b_2^{n+1} & \cdots & b_n^{n+1} & q_{n+1} \end{vmatrix} = 0$$

and we can write $\Delta_1 q_1 + \Delta_2 q_2 + \cdots + \Delta_{n+1} q_{n+1} = 0$, where $\{\Delta_i\}$ are determinants depending on $\{b_i\}$. One can continue and obtain the following set of equations

$$\begin{gathered} \Delta_1 q_1 + \Delta_2 q_2 + \cdots + \Delta_{n+1} q_{n+1} = 0, \\ \Delta_1 q_2 + \Delta_2 q_3 + \cdots + \Delta_{n+1} q_{n+2} = 0, \\ \cdots \\ \Delta_1 q_{n+1} + \Delta_2 q_{n+2} + \cdots + \Delta_{n+1} q_{2n+1} = 0. \end{gathered}$$

To have a nonzero solution for $\{\Delta_i\}$, condition (15) is necessary and sufficient. QED.

It is obvious that the condition of Proposition 2 is a particular case of the above condition, for $n=1$.

III. Expansion of Probability Density Function

We begin with some remarks concerning the probability density function (pdf) of a quadratic functional of a zero-mean vector using certain ideas of de Acosta [2]. If the dimension of the range of KT is zero, then with probability 1, $Q=0$. Let us consider the case of a real vector first. If $\dim[\mathscr{R}(KT)]=1$, then $Q=\lambda Z^2$, where Z is a real standard Gaussian variable. Then the pdf of Q is

$$f(t)=\exp(-t/2\lambda)\big/\sqrt{2\pi\lambda t}; \qquad \lambda>0, t>0, \quad (16)$$

which is an unbounded function for $t\to 0$. If $\dim[\mathscr{R}(KT)]=2$, then $Q=\lambda_1 Z_1^2+\lambda_2 Z_2^2$ and its pdf is

$$f(t)=\left(1/2\sqrt{\lambda_1\lambda_2}\right)\exp\left[-(t/4)\left(\lambda_1^{-1}+\lambda_2^{-1}\right)\right] \cdot I_0\left((t/4)\left(\lambda_1^{-1}+\lambda_2^{-1}\right)\right), \qquad t>0 \quad (17)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind [1], and is a bounded uniformly continuous function. If $3\le\dim[\mathscr{R}(KT)]\le\infty$, using bounds in Fourier analysis, one can demonstrate that the pdf is a bounded uniformly continuous function (Appendix C).

Let us consider now the case of a complex vector. If $\dim[\mathscr{R}(KT)]=1$, then $Q=\lambda|Z|^2$, where Z is a complex standard Gaussian variable, and its pdf is

$$f(t)=\exp(-t/\lambda)\big/\lambda; \qquad \lambda>0, t\ge 0, \quad (18)$$

which is a bounded uniformly continuous function. If $2\le\dim[\mathscr{R}(KT)]\le\infty$, this means at least a dimension 4 in the corresponding real space and then the pdf is a bounded uniformly continuous function.

We shall study the expansion of the pdf when it is a bounded uniformly continuous function, i.e. for all cases, except the case of a real vector with $\dim[\mathscr{R}(KT)]=1$, where the pdf is well known. An appropriate probability density function is chosen as initial approximation and an expansion in a series of functions is considered. This is a general method for approximating a probability distribution and particularly the distribution of a sum of random variables. As initial approximation we choose a pdf which is exact, in the case of a finite number of equal eigenvalues. This choice gives in the central case a gamma pdf

$$p_{a,b,0}(x)=x^a\exp(-x/b)\big/\left[b^{a+1}\Gamma(a+1)\right], \qquad x>0, \quad (19)$$

and in the noncentral case

$$p_{a,b,c}(x)=\left(x^{a/2}/bc^a\right)\exp\left(-(x+c^2)/b\right) \cdot I_a\left((2c\sqrt{x}/b)\right), \qquad x>0. \quad (20)$$

The parameters a, b, c can be chosen in order to have the two (or three if $c\neq 0$) first moments identical. The first expression (19) is given by Rice [26] as an approximation for the pdf of a nonnegative quadratic form in central case. In this case the parameters a and b have an interesting interpretation by the choice of gamma distribution as initial approximation: we have $2(a+1)$ "equivalent degrees of freedom" with $b/2$ "average variance per degree of freedom" for the quadratic form. These parameters are given by

$$a+1=\kappa_1^2/\kappa_2, \qquad b=\kappa_2/\kappa_1. \quad (21)$$

A. The Central Case

We propose the following series expression for the pdf $f(x)$:

$$f(x)=x^a\exp(-x/b)\big/\left[b^{a+1}\Gamma(a+1)\right]\sum_{n=0}^{\infty}c_n\mathscr{L}_n^{(a)}(x/\beta) \quad (22)$$

where $\mathscr{L}_n^{(a)}(\cdot)$ are the generalized Laguerre polynomials. The new parameter β is introduced to obtain the convergence of the series (22). The orthogonal polynomials for the pdf $p_{a,b,0}(x)$ in $[0,\infty)$ are $\mathscr{L}_n^{(a)}(x/b)$. But the corresponding series does not converge for all pdf $f(x)$. As we shall see, the necessary and sufficient condition for the series (22) to converge is

$$\beta^{-1}>2\left(b^{-1}-(\lambda_{\max})^{-1}\right) \qquad \text{for complex } X$$
$$\beta^{-1}>2\left(b^{-1}-(2\lambda_{\max})^{-1}\right) \qquad \text{for real } X. \quad (23)$$

This is a consequence of the equiconvergence theorem for Laguerre series [30]. In the above form this condition is given also by Calvez [6]. Thus one can take $\beta=b$, if $2b>\lambda_{\max}$, for a complex vector, and $b>\lambda_{\max}$, for a real vector. Knowing that in the complex case b represents an average of the eigenvalues $\{\lambda_i\}$ (cf. (5) and (21)), this condition indicates that the parameter β is needed, if $\lambda_{\max}$ dominates the other eigenvalues. (An example of a finite-dimensional vector follows: $\lambda_1=0.4$, $\lambda_2\div\lambda_{11}=0.06$; then $b=0.196<\lambda_1/2$.) Laguerre series have been used by other authors [11], [13], [14], [20] to expand the pdf of a quadratic functional of a Gaussian vector. The method of Kotz *et al.* [20] is applicable only to finite-dimensional vectors and with $a+1=n/2$, where n is the dimension of the vector. One cannot use this method for an infinite-dimensional vector. The general series presented by Gideon and Gurland [10] could include the series (22). But the choice of series parameters in [10] is such that only finite-dimensional vectors can be considered.

In what follows we determine the coefficients c_n and bounds on approximation error by truncation. We use a standard method similar to that of Kotz *et al.* [20] and Gideon and Gurland [10].

Determination of pdf Series' Coefficients: Since series (22) is converging uniformly by Laplace transform and using [9, (28), p. 174] for Laguerre polynomials, we obtain the moment-generating function

$$\Theta(s)=\frac{1}{(1+sb)^{a+1}}\sum_{n=0}^{\infty}c_n\binom{a+n}{n}z^n \quad (24)$$

with $z = 1 - b/\beta(1+sb)$. Using the expression

$$\frac{1}{(1+sb)^{a+1}} = \exp \sum_{n=1}^{\infty} \frac{\gamma_n(-sb)^n}{n!b^n}$$

with $\gamma_n = (n-1)!(a+1)b^n$, we obtain the relation

$$\Phi(z) = \exp \sum_{n=1}^{\infty} \frac{\kappa_n - \gamma_n}{n!b^n}\left(1 - \frac{b}{\beta(1-z)}\right)^n = \sum_{n=0}^{\infty} c_n \binom{a+n}{n} z^n. \tag{25}$$

Putting

$$A_0 = \sum_{n=1}^{\infty}\left(1 - \frac{b}{\beta}\right)^n \frac{\kappa_n - \gamma_n}{n!b^n},$$

$$A_n = \sum_{r=1}^{\infty} \frac{\kappa_r - \gamma_r}{r!b^r} \sum_{m=1}^{n} \binom{r}{m}(-b/\beta)^m \binom{m+n-1}{n}, \quad n \geq 1, \tag{26}$$

we have

$$\exp \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} c_n \binom{a+n}{n} z^n. \tag{27}$$

By derivation in z we obtain the recurrent relation

$$c_{n+1} = \frac{n!}{\Gamma(a+n+2)} \sum_{m=0}^{n} (m+1) \cdot \frac{\Gamma(a+n-m+1)}{(n-m)!} A_{m+1} c_{n-m}, \qquad n \geq 0,$$

$$c_0 = \exp A_0. \tag{28}$$

To summarize the procedure for determining the coefficients: we compute the cumulants (9), then the coefficients A_n of (26), and finally the c_n of (28).

If we can choose $\beta = b$ ($2b > \lambda_{\max}$ for a complex vector), the following simplification is possible:

$$A_0 = 0, \qquad c_0 = 1$$

$$A_n = \sum_{r=1}^{n} \frac{\kappa_r - \gamma_r}{r!b^r}(-1)^r \binom{n-1}{r-1}$$

and so the coefficients c_n require the knowledge of cumulants up to order n only.

One can also prove, using the expression of the cumulants γ_r, that

$$\sum_{r=1}^{n} \gamma_r \frac{(-1)^r}{r!b^r}\binom{n-1}{r-1} = -\frac{a+1}{n}.$$

Thus we obtain a further simplification of the expression for A_n, that is

$$A_n = \sum_{r=1}^{n} \frac{\kappa_r}{r!b^r}(-1)^r \binom{n-1}{r-1} + \frac{a+1}{n}.$$

Error Bounding: Here we study the error in approximating $f(x)$ by truncating the series (22). We use a standard method, also used by Kotz *et al.* [20]. We first search for a bound for the coefficients c_n. The sum in (24) must converge uniformly in the half-plane $1 + \lambda_{\max} \operatorname{Re}[s] > 0$ (for a complex vector). The region of convergence on the z-plane is a circle. The center of the circle in the z-plane is $z_0 = 1 - \delta/2$, with $\delta^{-1} = \beta(b^{-1} - \lambda_{\max}^{-1})$, and the radius is $r_0 = \delta/2$.

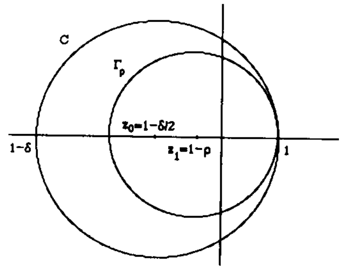

Fig. 1. Region of convergence of series (25).

To obtain the convergence of the coefficients c_n to zero, as $n \to \infty$, δ must be such that $1 - \delta/2 < 0$. Using Cauchy's integration formula, one can prove that the coefficients are bounded by

$$\binom{a+n}{n}|c_n| \leq \max_{\phi} |\Phi(z_0 + \rho e^{i\phi})| \epsilon^{n+1} \tag{29}$$

for $n > N(\epsilon)$ and with $1/(2\rho - 1) < \epsilon < 1$ and $1 < \rho < \delta/2$. In [20] the following inequality for Laguerre polynomials is given:

$$|\mathscr{L}_n^{(a)}(x)| \leq \exp(xR/(1+R)) \Big/ \left[(1-R)^{a+1}R^n\right] \tag{30}$$

for any R with $0 < R < 1$. The error of approximation in absolute value is given by

$$e_N(x) = x^a \exp(-x/b)/\left[b^{a+1}\Gamma(a+1)\right] \cdot \left|\sum_{n=N+1}^{\infty} c_n \mathscr{L}_n^{(a)}(x/\beta)\right|.$$

An upper bound using (30) can be expressed in terms of a hypergeometric function as follows:

$$e_N(x) \leq x^a \exp(-x/b)/\left[b^{a+1}\Gamma(a+1)\right] \cdot \inf_{D(\epsilon,\delta)} \left(\left[(1-R)^{a+1} R^{N+1} \binom{a+N+1}{N+1}\right]^{-1} \cdot \rho M_\Phi \epsilon^{N+2} \exp(xR/\beta(1+R))_2 \cdot F_1(1, N+1; a+N+1; \epsilon/R)\right) \tag{31}$$

where $D(\epsilon, \delta) = \{\rho, R\colon\ 1 < \rho < \delta/2,\ 1/(2\rho - 1) < \epsilon < R < 1\}$ and $M_\Phi = \sup_{0 \leq \phi \leq \infty} |\Phi(z_0 + \rho e^{i\phi})|$, for $1 < \rho < \delta/2$. It is obvious that the error converges to zero as $N \to \infty$.

Cumulative Distribution: Given the uniform convergence of the sum (22), we can integrate term-by-term and obtain the probability distribution

$$F(x) = \int_0^x f(y)\,dy = \frac{(\beta/b)^{a+1}}{\Gamma(a+1)} \sum_{n=0}^{\infty} c_n \int_0^{x/\beta} t^a \cdot \exp(-\beta t/b) \mathscr{L}_n^{(a)}(t)\,dt. \tag{32}$$

In Appendix B we calculate the integral

$$I(n,a)=\int_0^{x/\beta} t^a \exp(-\beta t/b)\mathscr{L}_n^{(a)}(t)\,dt.$$

We obtain, for $n\geq 1$,

$$I(n,a)=\exp(-x/\beta)\sum_{k=0}^{n-1}(x/\beta)^{a+k+1}\mathscr{L}_{n-k-1}^{(a+k+1)}(x/\beta)\cdot\frac{(-1)^n}{n!}\left(1-\frac{\beta}{b}\right)^k(n-k-1)!+\frac{(-1)^n}{n!}\left(1-\frac{\beta}{b}\right)^n I(0,a+n), \tag{33}$$

where

$$I(0,a+n)=\Gamma(a+n+1)(b/\beta)^{a+n+1}\cdot I_\Gamma\left(x/\sqrt{a+n+1},a+n\right)$$

and $I_\Gamma(\cdot,\cdot)$ is the incomplete gamma function, tabulated by Pearson [23]. If the choice $\beta=b$ is possible, then (33) is simplified in

$$I(n,a)=\frac{1}{n}\exp(-x/b)\left(\frac{x}{b}\right)^{a+1}\mathscr{L}_{n-1}^{(a+1)}\left(\frac{x}{b}\right),\qquad n\geq 1, \tag{34}$$

which is also given in [20] for a particular case. The truncation error is bounded by

$$\begin{aligned}E_N(x)&\leq(\beta/b)^{a+1}/\Gamma(a+1)\sum_{n=N+1}^{\infty}|c_n|\cdot\left|\int_0^{x/\beta}t^a\exp(-\beta t/b)\mathscr{L}_n^{(a+1)}(t)\,dt\right|\\&\leq(\beta/b)^{a+1}/\Gamma(a+1)\cdot\inf_{D(\epsilon,\delta)}\left[(1-R)^{a+1}(\beta/b-R/(1+R))^{a+1}\cdot R^{N+1}\binom{a+N+1}{N+1}\right]^{-1}\rho M_\Phi\epsilon^{N+2}I_{E2}\\&\quad\cdot F_1(1,N+1;a+N+1;\epsilon/R)\end{aligned} \tag{35}$$

with $I_E=I_\Gamma(x(\beta/b-R/(1+R))/(\beta\sqrt{a+1}),a)$ and $\beta/b>R/(1+R)$.

Remark 2: The method for expanding the probability distribution of $\Sigma_{i=1}^\infty\lambda_i|Z_i|^2$ in Laguerre polynomials is valid also for a sum of independent Gamma $\Gamma(a_i)$ variables $\Sigma_{i=1}^\infty b_iY_i$, with $b_i\geq 0$. The condition for the almost sure convergence of the above sum is $\Sigma_{i=1}^\infty(a_i+1)b_i<\infty$, and the cumulants of the series are, for $n=1,2,\cdots$,

$$\kappa_n=\sum_{i=1}^\infty(a_i+1)b_i^n.$$

B. The Noncentral Case

We propose to expand the pdf of Q in the noncentral case in a series of functions similar to (22) using the expression of the moment-generating function in (24). We can write this formula in the form

$$\Theta(s)=\frac{1}{(1+sb)^{a+1}}\sum_{n=0}^\infty c_n\binom{a+n}{n}\cdot\sum_{k=0}^n\binom{n}{k}(-b/\beta)^k\frac{1}{(1+sb)^k}.$$

By inverse Laplace transform we obtain, using the notation in (19),

$$f(x)=\sum_{n=0}^\infty c_n\binom{a+n}{n}\sum_{k=0}^n\binom{n}{k}(-b/\beta)^k p_{a+k,b,0}(x).$$

Thus in the noncentral case we propose the expansion of $f(x)$ in the series

$$f(x)=\sum_{n=0}^\infty c_n\binom{a+n}{n}\sum_{k=0}^n\binom{n}{k}(-b/\beta)^k p_{a+k,b,c}(x)=\sum_{k=0}^\infty d_k p_{a+k,b,c}(x). \tag{36}$$

The first term of the series (for $k=0$) is $p_{a,b,c}(x)$. This expansion is related to the noncentral chi-squared series in [20]. Again the difference is in the choice of series parameters; in [20], $a+1$ depends on the space dimension, which must be finite. By Laplace transform we obtain the moment-generating function (4 or 8)

$$\Theta(s)=\frac{\exp(-sc^2/(1+sb))}{(1+sb)^{a+1}}\sum_{n=0}^\infty c_n\binom{a+n}{n}\cdot\left(1-\frac{b}{\beta(1+sb)}\right)^n. \tag{37}$$

Then the coefficients c_n are evaluated by the same procedure as in (26)–(28), with κ_n as in (5 or 9) and

$$\gamma_n=(n-1)!(nc^2+a+1)b^n.$$

The coefficients c_n are again bounded, if (23) is valid, by the same expression (29) with the new function $\Phi(z)$ determined by the new cumulants.

To obtain a bound on the truncation error

$$e_n(x)=\left|\sum_{n=N+1}^\infty c_n\binom{a+n}{n}\varphi_n(x)\right|$$

we have to determine a bound of the functions in the expansion (36), that is, for the functions

$$\varphi_n(x)=\sum_{k=0}^n\binom{n}{k}(-b/\beta)^k p_{a+k,b,c}(x). \tag{38}$$

We give a bound for $\varphi_n(x)$ in Appendix A. The truncation error is bounded by

$$e_n(x)\leq p_{a,b,c}(x)\inf_{D(\epsilon,\delta)}\rho M_\Phi\epsilon^{N+2}/(1-\epsilon)+\inf_{D(\epsilon,\delta)}\epsilon\rho M_\Phi(\epsilon/R)^{N+1}\sigma_0(x;R)/(1-\epsilon/R). \tag{39}$$

In what follows we discuss the cumulative distribution $F(x)$. We integrate the series in (36) term-by-term and we obtain

$$F(x)=\sum_{n=0}^{\infty} c_n\binom{a+n}{n}\omega_n(x), \tag{40}$$

where

$$\omega_n(x)=\int_0^x \varphi_n(y)\,dy=\sum_{k=0}^{\infty}\binom{n}{k}(-b/\beta)^k Q_{a+k,b,c}(x) \tag{41}$$

and $Q_{a,b,c}(x)$ is the cumulative distribution (Marcum's function) of the probability density function $p_{a,b,c}(x)$. We give a bound for $\omega_n(x)$ in Appendix A. Using this bound and the bound for c_n we obtain the following expression for the truncation error

$$\begin{aligned}E_N(x)&=\left|\sum_{n=N+1}^{\infty} c_n\binom{a+n}{n}\omega_n(x)\right|\\ &\le Q_{a,b,c}(x)\inf_{D(\epsilon,\delta)}\epsilon\rho M_\Phi(\epsilon(b-\beta)/\beta)^{N+1}\\ &\quad/[1-\epsilon(b-\beta)/\beta]\\ &\quad+\inf_{D(\epsilon,\delta)}\epsilon\rho M_\Phi(\epsilon/R)^{N+1}\sigma_1(x;R)/[1-\epsilon/R]. \end{aligned}\tag{42}$$

This expression completes our method of approximation.

IV. Applications In Estimation and Detection Problems

Here we give some applications to detection and estimation problems. These examples illustrate the method proposed and its efficiency. We consider only the central case. The applications include the estimation of the average power

$$Q=\int_0^1 |X(t)|^2\,dt=\langle X,X\rangle$$

and certain problems of detection. Some theoretical statements concerning the above quadratic functional are given by Varberg [32]. We also give two examples of power spectrum estimation using the method of smoothed periodograms.

In the case of average power estimation the operator T of the quadratic functional is the identity. Thus we have $KT=K$. The detection problem is formulated as follows: H_1: X is a complex Gaussian process, with $K=K_s+K_n$; H_0: X is a complex Gaussian process, with $K=K_n$. It is well-known that, if $\mathscr{R}(K_s+K_n)=\mathscr{R}(K_n)$ and the operator $((K_s+K_n)^{-1}-K_n^{-1})K_n^{1/2}$ is defined in a dense subset of H and is Hilbert–Schmidt, then the sufficient statistics are given by [17], [25] $Q=\langle TX,X\rangle$, with T such that $K_s=(K_s+K_n)TK_n$. Thus, under hypothesis H_1 (evaluation of probability of detection), $KT=K_sK_n^{-1}$, which is a nuclear operator [25]. Under hypothesis H_0 (evaluation of false alarm probability), $KT=(I+K_sK_n^{-1})^{-1}K_sK_n^{-1}$, which is also a nuclear operator. The moment-generating function of Q is, for $1+\lambda_{\max}\operatorname{Re}[s]>0$ ($\lambda_{\max}$ being the largest eigenvalue of $K_sK_n^{-1}$),

$$\Theta_1(s)=\left(\det\left(I+sK_sK_n^{-1}\right)\right)^{-1} \qquad \text{under } H_1,$$

$$\Theta_0(s)=\left(\det\left(I+s\left(I+K_sK_n^{-1}\right)^{-1}K_sK_n^{-1}\right)\right)^{-1} \qquad \text{under } H_0.$$

Then we obtain

$$\begin{aligned}\Theta_0(s)&=\det\left(I+K_sK_n^{-1}\right)\left(\det\left(I+K_sK_n^{-1}+sK_sK_n^{-1}\right)\right)^{-1}\\ &=\det\left(I+K_sK_n^{-1}\right)\Theta_1(1+s).\end{aligned}$$

The probability density functions of Q under the two hypotheses are therefore related by

$$f_0(x)=\det\left(I+K_sK_n^{-1}\right)e^{-x}f_1(x).$$

This relation is also given in [21] for the finite-dimensional case. One can use the results given in this article to determine the probability of the two kinds of error. Which of $f_1(x)$ and $f_0(x)$ is easier to evaluate depends on whether it is easier to calculate the traces of KT under hypothesis H_1 or H_0. Apart from the likelihood criterion, the deflection criterion is also considered in signal detection [4]. For a random signal in independent Gaussian noise, the optimum operation T is defined by $T=K_n^{-1}K_sK_n^{-1}$, when certain conditions hold [4]. Then under hypothesis H_0, we have $KT=K_sK_n^{-1}$. We can therefore use the results given here to set the false alarm threshold. The following results are directly useful in detection problems, if the eigenvectors of the signal and the noise are the same and the eigenvalues of the signal are proportional to the square of those of the noise. (This is of course a strong, although plausible, assumption, which for stationary processes means that the power spectrum of the signal is approximately proportional to the square of the power spectrum of the noise.) Then $KT=K_n$ under hypothesis H_1.

Stationary Markov Process: The covariance function is normalized as in the following:

$$K(t,u)=\exp(-\mu|t-u|);\qquad 0\le t,u\le 1.$$

We give approximations of the probability density function, in Figs. 2 and 3, of the cumulative distribution of the

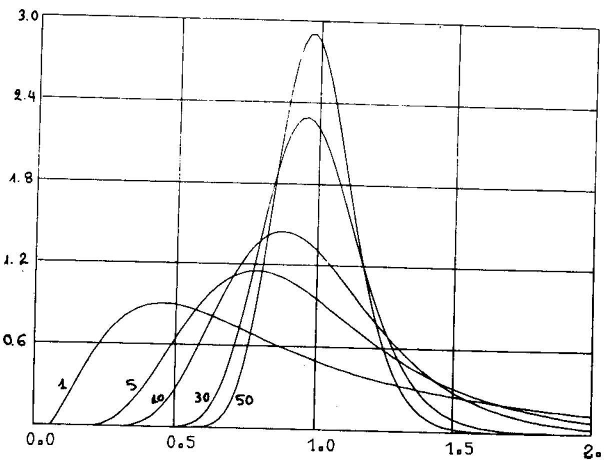

Fig. 2. Probability density function for complex stationary markov process ($\mu=1, 5, 10, 30, 50$).

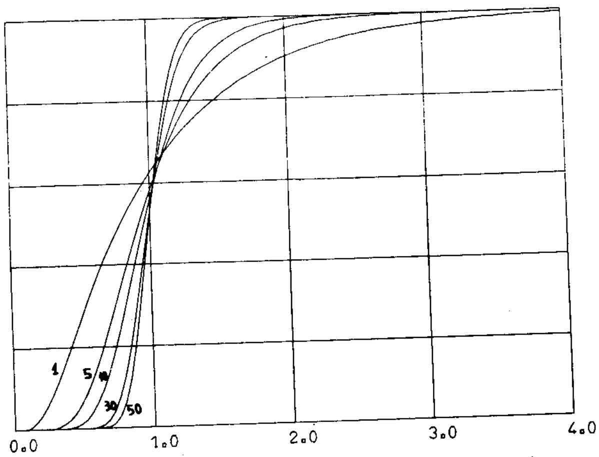

Fig. 3. Cumulative distribution for complex stationary markov process ($\mu = 1, 5, 10, 30, 50$).

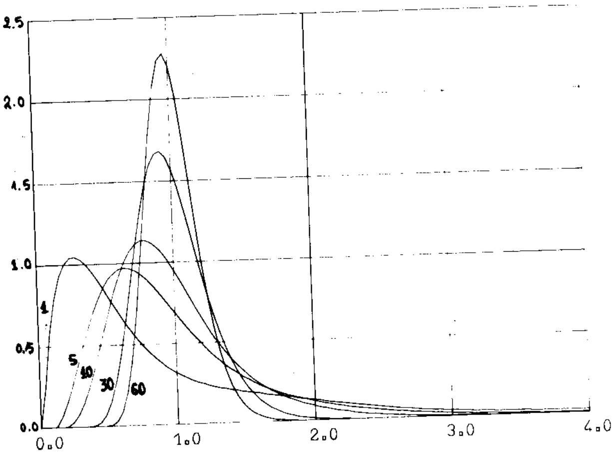

Fig. 5. Probability density function for real stationary markov process ($\mu = 1, 5, 10, 30, 60$).

average power for $\mu = 1$, 5, 10, 30, 50 for the case of complex variables. We have calculated the cumulants using a recurrence formula [31]. We give here the three first cumulants given also in [21]:

$$\kappa_1 = 1,$$
$$\kappa_2 = \left(2\mu + e^{-2\mu} - 1\right)/\left(2\mu^2\right),$$
$$\kappa_3 = 3\left[(\mu - 1) + (\mu + 1)e^{-2\mu}\right]\mu^{-3}.$$

One can see that for $\mu \to 0$, the parameters $a + 1$ and b both tend to 1. Also, for large μ, $a + 1$ is asymptotically like $\mu + 0.5$. On the other hand, $q_3/q_2 \to 1$, for $\mu \to 0$, and it is asymptotically like $3/(2\mu + 1)$ for large μ. This means that for $\mu \to 0$ we obtain a χ^2 distributed random variable with 2 degrees of freedom and variance 0.5 per degree of freedom.

In Fig. 4 we illustrate the successive approximations of the probability density function over a portion of the range for $\mu = 5$. We give Rice's approximation and those using $N_c = 3 - 8$ (N_c: number of coefficients). We have observed that as μ grows, the convergence is reached with less coefficients.

The case of real variables is studied by Slepian [28] and Grenander *et al.* [12] using two different methods. We

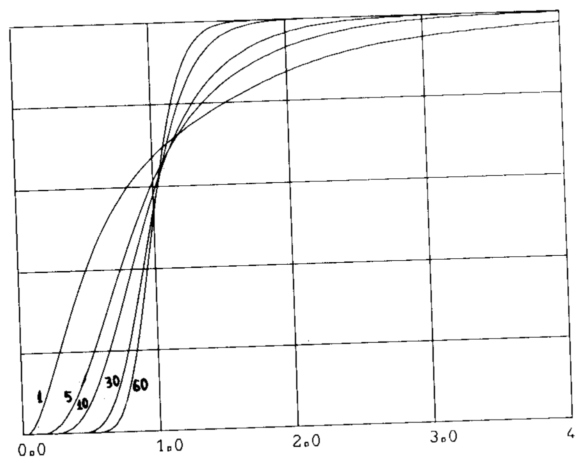

Fig. 6. Cumulative distribution for real stationary markov process ($\mu =$ 1, 5, 10, 30, 60).

have applied our method in this case and we give in Figs. 5 and 6 the probability density function and the cumulative distribution, respectively. These results as well as those in [28] are fairly accurate approximations of the probability distribution.

Brownian Motion: The covariance function is

$$K(t,u) = \min(t,u); \qquad 0 \le t, u \le 1.$$

For complex variables we give in Fig. 7 the exact cumulative distribution of the average power and the approximation using our method with 20 coefficients. One can see that the two curves coincide for the precision of the figure. For complex variables we give in Fig. 8 the exact probability density function of the average power and the approximation using our method ($N_c = 20$).

The exact cumulative distribution is obtained from the moment-generating function. Knowing that $\lambda_i^{-1} = (i - 0.5)^2\pi^2$, we obtain [1]

$$\Theta(s) = \prod_{i=1}^{\infty} \left(1/(1 + s\lambda_i)\right) = \left(1/\cosh\sqrt{s}\right). \tag{43}$$

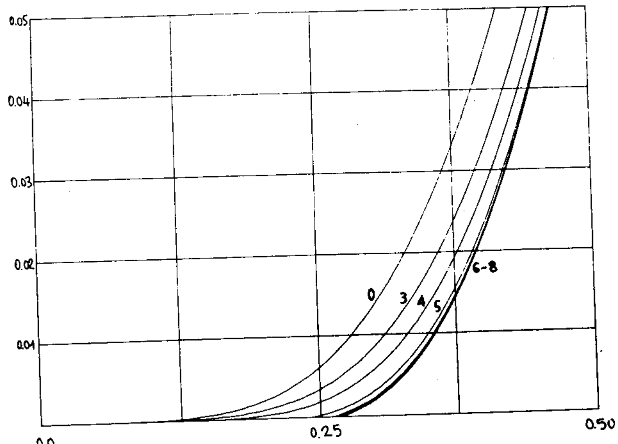

Fig. 4. Approximations of cumulative distribution for complex stationary markov process ($\mu = 5$; $N_c = 0, 3$–8).

This expression was given by Dugué [8] for the case of a

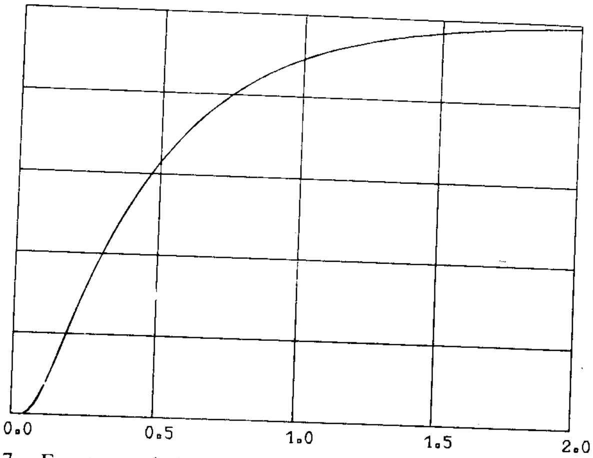

Fig. 7. Exact cumulative distribution and our approximation ($N_c = 20$) for complex Brownian motion.

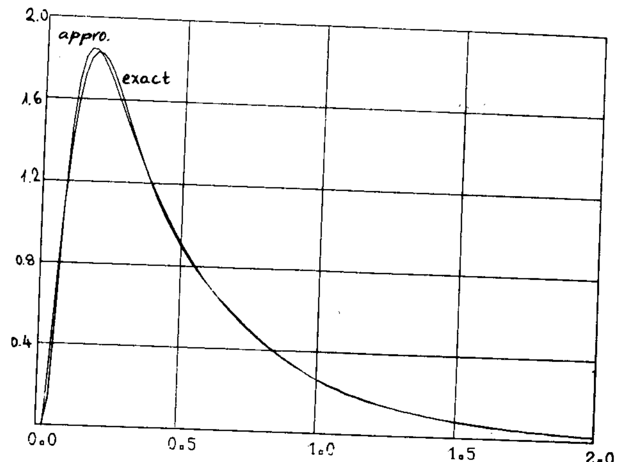

Fig. 8. Exact probability density function and our approximation ($N_c = 20$) for complex Brownian motion.

real process. By inverse Laplace transform [9], (43) leads to

$$F(x) = 2 \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc}\left((n+0.5)/\sqrt{x}\right),$$

where erfc is the complement of the error function.

Brownian "Bridge": The covariance function is as follows:

$$K(t,u) = \min(t,u) - tu; \qquad 0 \le t, u \le 1.$$

The real version of the quadratic form in this case is the limit of the von Mises statistics used in certain "goodness of fit" criteria [3]. For complex variables we give in Fig. 9 the exact cumulative distribution of the average power and the approximation using ten coefficients. The exact expression is obtained as in the preceding case. The moment-generating function is evaluated knowing that $\lambda_i = 1/(i\pi)^2$ [3] and using the expansion of sinh [1]

$$\Theta(s) = \prod_{i=1}^{\infty} (1/1 + s\lambda_i) = \sqrt{s}/\sinh\sqrt{s}. \qquad (44)$$

This expression was given by Smirnov [29] for the case of a real process. By inverse Laplace transform [9], (44) leads to

$$F(x) = \left(2/\sqrt{\pi x}\right) \sum_{n=0}^{\infty} \exp\left(-(n+0.5)^2/x\right).$$

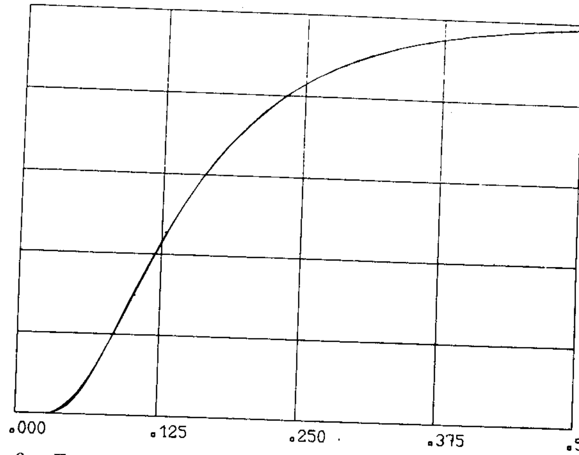

Fig. 9. Exact cumulative distribution and our approximation ($N_c = 10$) for complex Brownian "bridge."

For complex variables we also give in Fig. 10 the exact probability density function, the approximation using our method with 14 coefficients, and Rice's approximation.

For real variables we give in Fig. 11 the probability density function and the cumulative distribution ($N_c = 14$). This case has been studied by Anderson and Darling [3] who give an expression of the cumulative distribution

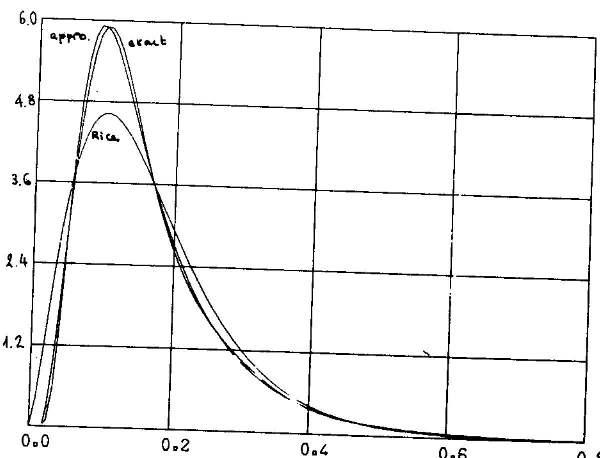

Fig. 10. Exact probability density function, Rice's and our approximation ($N_c = 14$) for complex Brownian "bridge."

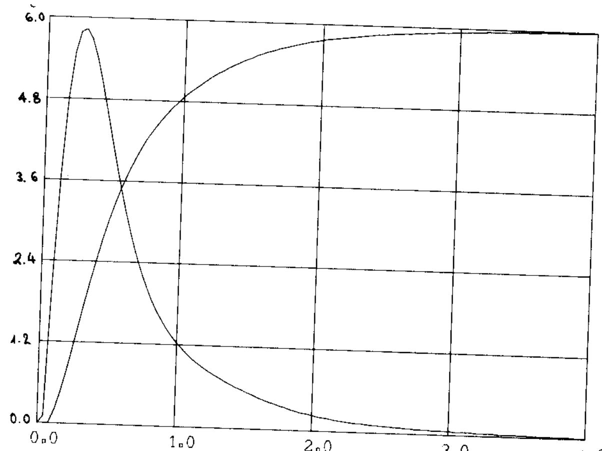

Fig. 11. Probability density function and cumulative distribution ($N_c = 14$) for real Brownian "bridge."

using the modified Bessel function. The numerical table in [3] confirms the efficiency of our method.

Power Spectrum Estimation: The smoothed periodogram as an estimator of the power spectrum is a positive-definite quadratic form of the process. The properties of quadratic forms can be used to study this estimator. We have applied our method to evaluate the probability distribution of the smoothed periodogram of a time series at frequency 0, that is,

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i-j} X_i X_j.$$

We study Q for a real time series with covariance given by

$$E\{X_i X_j\} = \rho^{|i-j|}, \qquad 0 \le \rho \le 1.$$

We consider two different windows:

a) Hanning window

$$w_{i-j} = N^{-1}\cos^2\left((i-j)\pi/2M_1\right); \qquad |i-j| < M_1,$$
$$= 0; \qquad |i-j| \ge M_1,$$

b) sinc^2 window

$$w_{i-j} = N^{-1}\,\mathrm{sinc}^2\left((i-j)\pi/M_2\right).$$

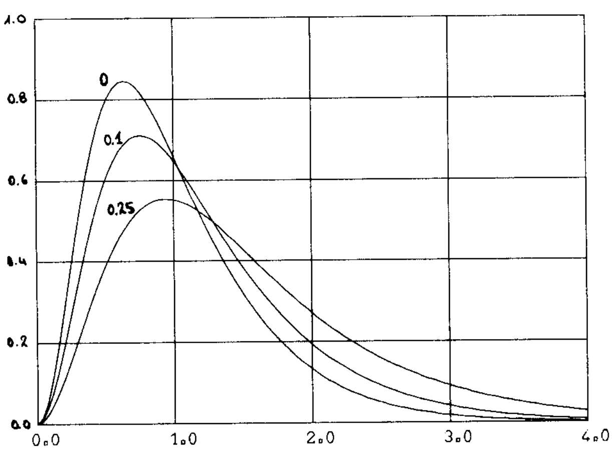

Fig. 12. Probability density function for smoothed periodogram using Hanning window ($\rho = 0, 0.1, 0.25$).

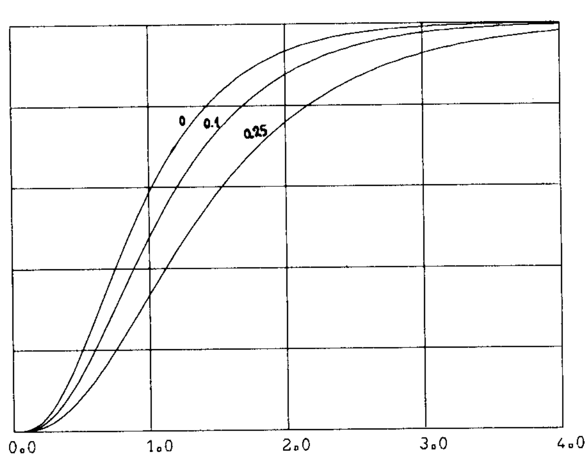

Fig. 13. Cumulative distribution for smoothed periodogram using Hanning window ($\rho = 0, 0.1, 0.25$).

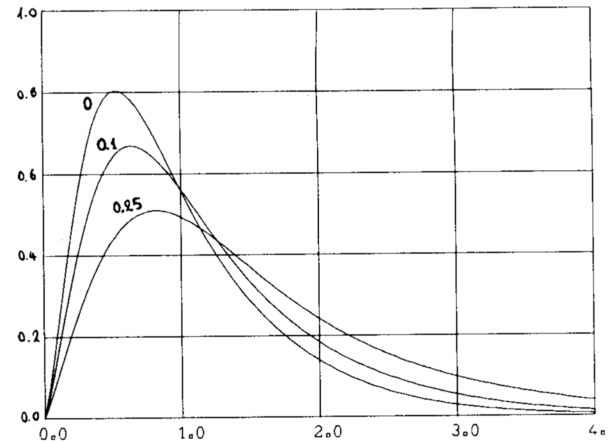

Fig. 14. Probability density function for smoothed periodogram using sinc^2 window ($\rho = 0, 0.1, 0.25$).

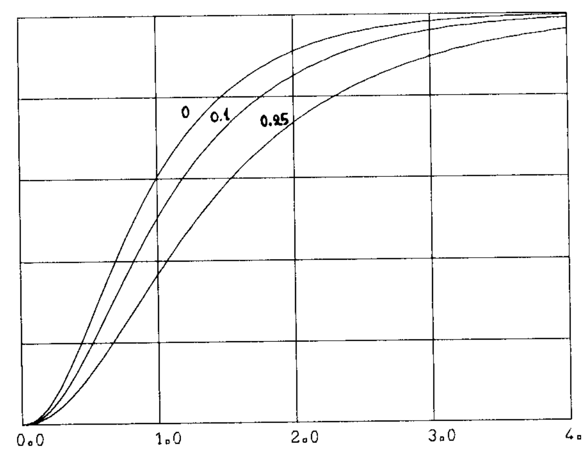

Fig. 15. Cumulative distribution for smoothed periodogram using sinc^2 window ($\rho = 0, 0.1, 0.25$).

We give numerical results for $N = 20$, $M_1 = 5$, $M_2 = 8$, and three values of $\rho = 0, 0.1, 0.25$. For the Hanning window these results are given in Figs. 12 and 13 and for the sinc^2 window in Figs. 14 and 15. The last case is also studied in [12]. Our results are close to the fairly accurate curves given in [12].

The cases presented here illustrate that our method is efficient and gives correct results. The criterion of minimum quadratic error over the entire domain of the pdf requires many coefficients to obtain a good approximation at all points. One could use a different number of coefficients at each point, since the series converges at all points. We have not tried to apply this idea, but we have remarked that the convergence is slower for small values of the pdf because the weight function in the quadratic error of approximation is more important for these values. Thus, we have used a minimax-like criterion for the number of coefficients in the most unfavorable case. We have used a maximum of 20 coefficients to obtain a good approximation of probabilities.

V. Summary

In this paper, we use only traces of the operators that define the quadratic form, not eigenvalues. This yields two advantages: 1) the properties of the form are better used, because the traces use all eigenvalues in a more concise way, and 2) in certain specific cases it is more difficult to compute the eigenvalues than to compute the traces numerically. Our study is complete for the case of positive-definite quadratic forms in Gaussian variables. That is, we give algorithms for evaluating the coefficients, conditions for convergence of the series, and a bound for the error of approximation by truncation. The applications given here are chosen to illustrate the method and to compare with results of other authors.

Acknowledgment

The author wishes to thank Professor C. R. Baker for many helpful discussions, and Professor S. Cambanis for suggesting Proposition 3. He is also grateful to a reviewer for suggesting a simpler proof of Proposition 2.

Appendix A
Bounding a Series of Functions

Let us consider a series of functions $\{f_n(x)\}$. We introduce the generating function of the series

$$G(t,x)=\sum_{n=0}^{\infty} t^n f_n(x). \tag{A1}$$

If the sum exists for $|t|<T$, we can write, using Cauchy's integral formula, for $n=0,1,2,\cdots$,

$$\begin{aligned} f_n(x) &= (1/2\pi i)\int_{\Gamma_R} G(t,x)/t^{n+1}\,dt \\ &= (1/2\pi)\int_0^{2\pi} G(\mathrm{Re}^{i\theta},x)/R^n e^{in\theta}\,d\theta \end{aligned} \tag{A2}$$

where Γ_R is the circle $|t|=R<T$. We can bound this function by

$$|f_n(x)| \le \inf_{0<R<T} \frac{1}{R^n} \sup_{0\le\theta\le 2\pi} |G(\mathrm{Re}^{i\theta},x)|. \tag{A3}$$

In what follows we apply this method for the series of functions in expansion (36) and also in expansion (40). The functions in the first case (36) are given by (38), and then the generating function for $|t|<1$ is

$$\begin{aligned} G_0(t,x) &= \sum_{n=0}^{\infty} t^n \varphi_n(x) \\ &= (x^{a/2}/bc^a)\exp(-(x+c^2)/b) \\ &\quad\cdot \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}\left(-b\sqrt{x}/(\beta c)\right)^k I_{a+k}(2c\sqrt{x}/b) \\ &= [x^{a/2}/bc^a(1-t)]\exp(-(x+c^2)/b) \\ &\quad\cdot \sum_{n=0}^{\infty}\left(-b\sqrt{x}\,t/[\beta c(1-t)]\right)^n I_{a+n}((2c\sqrt{x}/b)). \end{aligned} \tag{A4}$$

We can evaluate this sum in a simpler way in the complex plane after Laplace transform

$$\begin{aligned} g_0(t,s) &= (1-t)^{-1}\sum_{n=0}^{\infty}\left(-bt/[\beta(1-t)]\right)^n(1+sb)^{-(a+n+1)} \\ &\quad\cdot\exp(-sc^2/(1+sb)) \\ &= (1-t)^{-1}\frac{\exp(-sc^2/(1+sb))}{(1+sb)^{a+1}} \\ &\quad\cdot\left(1-bt/[\beta(1-t)+bt+sb\beta(1-t)]\right). \end{aligned} \tag{A5}$$

By inverse Laplace transform we obtain

$$\begin{aligned} G_0(t,x) &= (1-t)^{-1}p_{a,b,c}(x) - t[\beta(1-t)]^{-1}p_{a,b,c}(x) \\ &\quad * \exp\left(-x\left(b^{-1}+t[\beta(1-t)]^{-1}\right)\right), \end{aligned} \tag{A6}$$

where $*$ means the convolution. For the first term and for all $n=0,1,2,\cdots$, we have

$$(2\pi i)^{-1}\int_{\Gamma_R}\left[(1-t)t^{n+1}\right]^{-1}dt=1, \qquad 0<R<1.$$

Then for the bound of functions $\{\varphi_n(x)\}$ we can write

$$|\varphi_n(x)| \le p_{a,b,c}(x) + \inf_{0<R<1}\sigma_0(x;R)/R^n, \tag{A7}$$

where

$$\begin{aligned} \sigma_0(x;R) &= R[\beta(1-R)]^{-1}p_{a,b,c}(x) \\ &\quad * \exp\left(-x\left(b^{-1}-R[\beta(1+R)]^{-1}\right)\right) \\ &= [\beta(1-R)]^{-1}\exp\left(-x\left(b^{-1}-R[\beta(1+R)]^{-1}\right)\right. \\ &\quad \left. -c^2[b]^{-1}\left(1-\beta R[b(1+R)]^{-1}\right)\right) \\ &\quad\cdot\left(\beta(1+R)/bR\right)^{a+1}Q_{a,\beta(1+R)/R,\,c\beta(1+R)/bR}(x), \end{aligned}$$

where $Q_{a,b,c}(x)$ is the cumulative distribution (Marcum's function) for the probability density function $p_{a,b,c}(x)$.

We also apply this method for bounding functions $\{\omega_n(x)\}$ in (41). In this case the Laplace transform of the generating function of $\{\omega_n(x)\}$ is

$$g_1(t,s)=g_0(t,s)/s. \tag{A8}$$

Using the same method as above we obtain the bound of functions $\{\omega_n(x)\}$:

$$|\omega_n(x)| \le [(b-\beta)/\beta]^n Q_{a,b,c}(x) + \inf_{0<R<1}\sigma_1(x;R)/R^n, \tag{A9}$$

where $\sigma_1(x;R)$ is determined as in the preceding derivation.

Appendix B
Certain Integrals of Laguerre Polynomials

We want calculate the integral in (32). Using, for $n\ge 1$,

$$t^a e^{-t}\mathscr{L}_n^{(a)}(t)\,dt = d\left(t^{a+1}e^{-t}\mathscr{L}_{n-1}^{(a+1)}(t)\right)/n, \tag{B1}$$

we obtain

$$\begin{aligned} I(n,a) &= n^{-1}(x/\beta)^{a+1}\exp(-x/\beta)\mathscr{L}_{n-1}^{(a+1)}(x/\beta) \\ &\quad - n^{-1}I(n-1,a+1). \end{aligned} \tag{B2}$$

We apply this relation n times and we have, for $k=0,1,\cdots,n-1$,

$$I(n-k,a+k)=(n-k)^{-1}(x/\beta)^{a+k+1}\exp(-x/\beta)\cdot\mathscr{L}_{n-k-1}^{(a+k+1)}(x/\beta)-(n-k)^{-1}(1-\beta/b)\cdot I(n-k-1,a+k+1) \tag{B3}$$

and thus

$$\sum_{k=0}^{n-1}(-1)^k(n-k)!(1-\beta/b)^k I(n-k,a+k)$$
$$=(x/\beta)^{a+1}\exp(-x/\beta)\sum_{k=0}^{n-1}(-1)^k(n-k-1)!\cdot(x/\beta(1-\beta/b))^k\mathscr{L}_{n-k-1}^{(a+k+1)}(x/\beta)+\sum_{k=0}^{n-1}(-1)^{k+1}(n-k-1)!(1-\beta/b)^{k+1}\cdot I(n-k-1,a+k+1),$$

from which (33) follows.

Appendix C
Bounding the PDF for $\dim[\mathscr{R}(KT)]\geq 3$

One can write, from (8),

$$f(x)=(1/2\pi)\int_{-\infty}^{\infty}\left[\pi\sqrt{1+2j\omega\lambda_i}\right]^{-1}\exp(-j\omega x)\,d\omega.$$

If $\lambda_1>\lambda_2>\lambda_3>\cdots$, we have

$$f(x)\leq\frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{i=1,2,3}\left(1+4\omega^2\lambda_i^2\right)^{-1/4}d\omega$$
$$\leq\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1+4\omega^2\lambda_3^2\right)^{-3/4}d\omega.$$

For all $h>0$, we obtain

$$f(x)\leq\frac{1}{2\pi}\left[\int_{-h}^{h}\left(1+4\omega^2\lambda_3^2\right)^{-3/4}d\omega+2\int_{h}^{\infty}\left(1+4\omega^2\lambda_3^2\right)^{-3/4}d\omega\right]$$
$$\leq\frac{1}{\pi}\left[h+\int_{h}^{\infty}(2\omega\lambda_3)^{3/2}d\omega\right]=\frac{1}{\pi}\left(h+\left(2h\lambda_3^3\right)^{-1/2}\right).$$

This last function is minimized for $h=1/(2\lambda_3)$. Finally $f(x)$ can be bounded by $f(x)\leq 3/(2\pi\lambda_3)$. Similar arguments can be used to demonstrate that $f(x)$ is uniformly continuous.

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