Collective dynamics of 'small-world' networks

Duncan J. Watts & Steven H. Strogatz
Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, New York 14853, USA

ABSTRACT

Networks of coupled dynamical systems have been used to model biological oscillators, Josephson junction arrays, excitable media, neural networks, spatial games, genetic control networks and many other self-organizing systems. Ordinarily, the connection topology is assumed to be either **completely regular or completely random**. But many biological, technological and social networks lie somewhere **between these two extremes**.

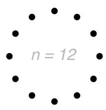
Here we explore simple models of networks that can be tuned through this middle ground: **regular networks 'rewired'** to introduce increasing amounts of disorder. We find that these systems can be highly clustered, like regular lattices, yet have small characteristic path lengths, like random graphs. We call them **'small-world' networks**, by analogy with the small-world phenomenon (popularly known as six degrees of separation). The neural network of the worm *Caenorhabditis elegans*, the power grid of the western United States, and the collaboration graph of film actors are shown to be small-world networks.

Models of dynamical systems with small-world coupling display enhanced signal-propagation speed, computational power, and synchronizability. In particular, infectious diseases spread more easily in small-world networks than in regular lattices.

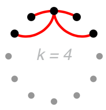
ALGORITHM

To interpolate between regular and random networks, we consider the following random rewiring procedure.

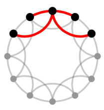
We start with a ring of n vertices



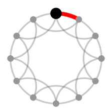
where each vertex is connected to its k nearest neighbors



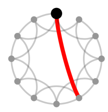
like so.



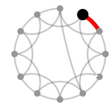
We choose a vertex, and the edge to its nearest clockwise neighbour.



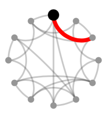
With probability p, we reconnect this edge to a vertex chosen uniformly at random over the entire ring, with duplicate edges forbidden. Otherwise, we leave the edge in place.



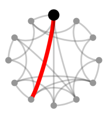
We repeat this process by moving clockwise around the ring, considering each vertex in turn until one lap is completed.



Next, we consider the edges that connect vertices to their second-nearest neighbours clockwise.



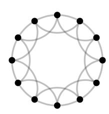
As before, we randomly rewire each of these edges with probability p.



We continue this process, circulating around the ring and proceeding outward to more distant neighbours after each lap, until each original edge has been considered once.

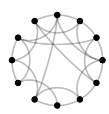
As there are $nk/2$ edges in the entire graph, the rewiring process stops after $k/2$ laps.

For $p = 0$, the ring is unchanged.

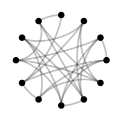


As p increases, the graph becomes increasingly disordered.

p=0.15



At $p = 1$, all edges are re-wired randomly.

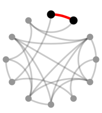


This construction allows us to 'tune' the graph between regularity ($p = 0$) and disorder ($p = 1$), and thereby to probe the intermediate region $0 < p < 1$, about which little is known.

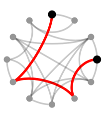
METRICS

We quantify the structural properties of these graphs by their **characteristic path length *L(p)*** and **clustering coefficient *C(p)***.
L(p) measures the typical separation between two vertices (a global property). *C(p)* measures the cliquishness of a typical neighbourhood (a local property).

L is defined as the number of edges in the shortest path between two vertices

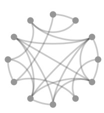


shortest path is 1 edge

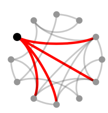


shortest path is 3 edges

averaged over all pairs of vertices.

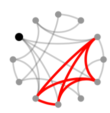


C is defined as follows. Suppose that a vertex v has k_v neighbours.



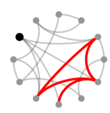
$k_v = 4$ neighbours

Then at most $k_v (k_v - 1) / 2$ edges can exist between them. (This occurs when every neighbor of v is connected to every other neighbour of v.)



at most 6 edges between 4 neighbours

Let C_v denote the fraction of these allowable edges that actually exist. Define C as the average of C_v over all vertices.

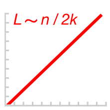


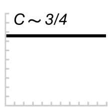
4 out of 6 edges exist. $C_v = 4/6 = 0.67$

For friendship networks, these statistics have intuitive meanings: L is the average number of friendships in the shortest chain connecting two people. C_v reflects the extent to which friends of v are also friends of each other; and thus C measures the cliquishness of a typical friendship circle.

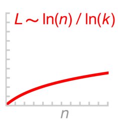
SMALL WORLDS

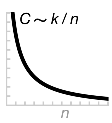
The regular lattice at $p = 0$ is a highly clustered, large world where L grows linearly with n.



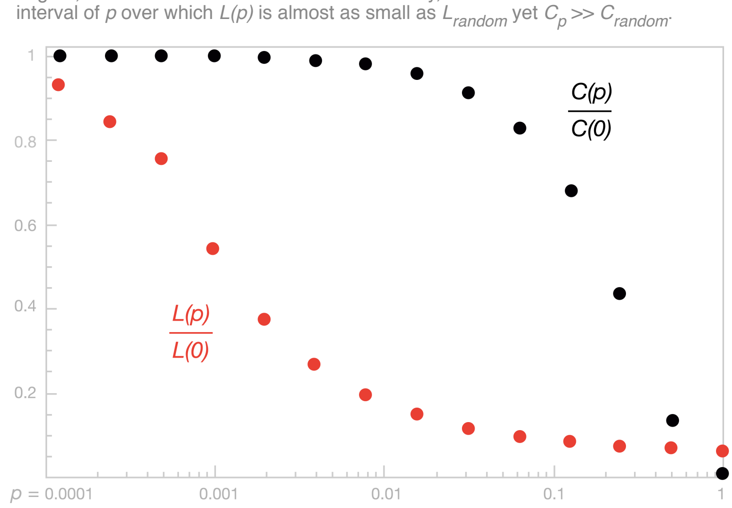


The random network at $p = 1$ is a poorly clustered, small world where L grows only logarithmically with n.



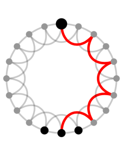


These limiting cases might lead one to suspect that large C is always associated with large L, and small C with small L. On the contrary, we find that there is a broad interval of p over which $L(p)$ is almost as small as L_{random} yet $C_p >> C_{random}$.

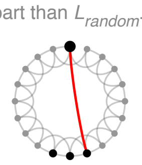


The data shown in the figure are averages over 20 random realizations of the rewiring process, and have been normalized by the values L(0), C(0) for a regular lattice. All the graphs have n = 1000 vertices and an average degree of k = 10 edges per vertex. We note that a logarithmic horizontal scale has been used to resolve the rapid drop in L(p), corresponding to the onset of the small-world phenomenon. During this drop, C(p) remains almost constant at its value for the regular lattice, indicating that the transition to a small world is almost undetectable at the local level.

These small-world networks result from the immediate drop in *L(p)* caused by the introduction of a few long-range edges. Such 'short cuts' connect vertices that would otherwise be much farther apart than L_{random}. For small p, each short cut has a highly nonlinear effect on L, contracting the distance not just between the pair of vertices that it connects, but between their immediate neighbourhoods, neighbourhoods of neighbourhoods and so on.

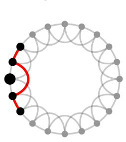


5 hops to neighbourhood

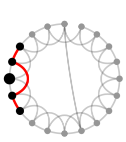


shortcut to neighbourhood

By contrast, an edge removed from a clustered neighbourhood to make a short cut has, at most, a linear effect on C; hence *C(p)* remains practically unchanged for small p even though *L(p)* drops rapidly. The important implication here is that at the local level (as reflected by *C(p)*), the transition to a small world is almost undetectable.



The 4 neighbors of each vertex have 3 out of 6 edges among themselves. C = 3/6 = 0.5



With shortcut, this is still true for almost every vertex. C = 0.48