Linear Prediction.

1

FLP given M previous values $[u(n-1), \dots u(n-M)]$
(predict)
estimate $u(n)$

$$\underline{u}_M(n-1) = \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(n-M) \end{bmatrix}$$

FLP error : $f_M(n) = u(n) - \sum_{k=1}^{M} w_{f,k}^{*} u(n-k) = u(n) - \underline{w}_f^H \cdot \underline{u}_M(n-1)$

Στοχος : Να βρεθεί το $\underline{w}_f$ τ.ω. $E\left[|f_M(n)|^2\right] \rightarrow \min$

Λυση WHE $\quad \underline{\underline{R}}_M \cdot \underline{w}_f = \underline{r}_M \quad$ (1) $\quad \Rightarrow \underline{w}_f = \underline{\underline{R}}_M^{-1} \underline{r}_M$

οπου
$$\underline{\underline{R}}_M = \begin{bmatrix} r(0) & r(1) & \cdots & r(M-1) \\ r^*(1) & & & \\ \vdots & & & \\ r^*(M-1) & & & r(0) \end{bmatrix}_{M\times M}, \quad \underline{w}_f = \begin{bmatrix} w_{f,1} \\ w_{f,2} \\ \vdots \\ w_{f,M} \end{bmatrix}_{M\times 1}, \quad \underline{r}_M = \begin{bmatrix} r^*(1) \\ r^*(2) \\ \vdots \\ r^*(M) \end{bmatrix}_{M\times 1}$$

Ισχυς του σφαλματος, P_M : Mean-squared Error.

$$\boxed{P_M = E\left[|f_M(n)|^2\right] \overset{\text{απο WHE}}{=} r(0) - \underline{r}_M^H \underline{w}_f = r(0) - \underline{r}_M^H \underline{\underline{R}}_M^{-1} \underline{r}_M} \quad (2)$$

$$P_M = E\left[|f_M(n)|^2\right] \overset{\text{απο WHE}}{=} \sigma_d^2 - \underline{p}^H \underline{w}_f = r(0) - \underline{r}_M^H \underline{w}_f = r(0) - \underline{r}_M^H \underline{\underline{R}}_M^{-1} \underline{r}_M$$

$\sigma_d^2 = r(0)$ $\qquad$ $\underline{w}_f$ απο WH $\rightarrow \underline{w}_f = \underline{\underline{R}}_M^{-1} \underline{r}_M$

$d(n) = u(n) \Rightarrow \sigma_d^2 = r(0)$ $\quad$ $u(n)$ is zero mean

$$\underline{p}^H = E\left[\underline{u}_M(n-1)\, u^*(n)\right] = \underline{r}_M$$

Augmented WHE for FLP

$$
① \,\&\, ② \longrightarrow \underbrace{\begin{bmatrix} r(0) & \underline{r}^H \\ \underline{r} & \underline{\underline{R}}_M \end{bmatrix}}_{\underline{\underline{R}}_{M+1}} \underbrace{\begin{bmatrix} 1 \\ -\underline{w}_f \end{bmatrix}}_{\underline{a}_M} = \begin{bmatrix} P_M \\ \underline{0} \end{bmatrix}
$$

όπου $a_{M,k} = \begin{cases} 1 & \text{για } k=0 \\ -w_{f,k} & \text{για } k=1,\ldots,M \end{cases}$

→ forward prediction error filter

Το διάνυσμα $\underline{a}_M$ έχει τους συντελεστές του φίλτρου σφάλματος FP

$$
f_M(n) = u(n) - \underline{w}_f^H \underline{u}(n-1) = 1\, u(n) - w_{f,1}^* u(n-1) - \ldots - w_{f,M}^* u(n-M)
$$

$$
= \underline{a}_M^H \, \underline{u}_{M+1}(n)
$$

BLP given M future values $[u(n), u(n-1), \ldots, u(n-M+1)] = \underline{u}_M(n)$

estimate the value $u(n-M)$

$$
\underline{u}_M(n) = \begin{bmatrix} u(n) \\ u(n-1) \\ u(n-2) \\ \vdots \\ u(n-M+1) \end{bmatrix}
$$

BLP error: $b_M(n) = u(n-M) - \sum_{k=1}^{M} w_{b,k}^* u(n-k+1)$

Στόχος: Να βρεθεί το $\underline{w}_b$ τ.ω. $E\left[|b_M(n)|^2\right] \to \min$

③

<u>λύση</u>

WHE : $\underline{\underline{R}} \cdot \underline{w} = \underline{P}$

WH		BP
$\underline{\underline{R}}$	$\longrightarrow$	$\underline{\underline{R}} = E[\underline{u}(n)\,\underline{u}^H(n)] = \begin{bmatrix} r(0) & r(1) & \cdots & r(M-1) \\ r^*(1) & & & \\ \vdots & & & \\ r^*(M-1) & & & r(0) \end{bmatrix}$
$\underline{w}$	$\longrightarrow$	$\underline{w}_b = [w_{b,1}, w_{b,2}, \ \ w_{b,M}]^T$

$$\underline{P} = E[\underline{u}(n) \cdot d^*(n)] \longrightarrow E[\underline{u}(n) \cdot u^*(n-M)] = \begin{bmatrix} r(M) \\ r(M-1) \\ \vdots \\ r(1) \end{bmatrix} = \underline{r}_M^{*B}$$

άρα η λύση $\longrightarrow$ $\underline{\underline{R}}\ \underline{w}_b = \underline{r}^{*B}$

από WHE : $E[|b_M(n)|^2] = \sigma_d^2 - \underline{P}^H \underline{w}_b = r(0) - \underline{r}_M^{BT} \underline{w}_b$

Augmented WH Eq for BLP

$\underline{0} \Rightarrow M\times M$

$$\underbrace{\begin{bmatrix} \underline{\underline{R}} & \underline{r}^{B*} \\ \underline{r}^{BT} & r(0) \end{bmatrix}}_{\underline{\underline{R}}_{M+1}} \underbrace{\begin{bmatrix} -\underline{w}_b \\ 1 \end{bmatrix}}_{\underline{c}_M} = \begin{bmatrix} \underline{0} \\ P_M \end{bmatrix}$$

όπου $c_{M,k} = \begin{cases} -w_{b,k+1} & k = 0, 1, \ldots, M-1 \\ 1 & k = M \end{cases}$

Το διάνυσμα $\underline{c}_M$ είναι το backward prediction error filter

$$b_M(n) = u(n-M) - \underline{w}_b^H \underline{u}(n) = \sum_{k=0}^{M} c_{M,k}^* u(n-k)$$

Σχέση μεταξύ FLP & BLP

α. σχέση μεταξύ των φίλτρων

FLP: $\underline{\underline{R}} \cdot \underline{w}_f = \underline{r}$

BLP: $\underline{\underline{R}} \cdot \underline{w}_b = \underline{r}^{B*} \Leftrightarrow E\left[\underline{u}(n) \cdot \underline{u}^H(n) \cdot \underline{w}_b\right] = \underline{r}^{B*} \Leftrightarrow$

$$E\left[\underline{u}(n) \cdot \underline{u}^{BH}(n)\, \underline{w}_b^B\right] = \underline{r}^{B*} \Leftrightarrow$$

$$\left(E\left[\underline{u}(n) \cdot \underline{u}^{BH}(n)\, \underline{w}_b^B\right]\right)^B = \left(\underline{r}^{B*}\right)^B \Leftrightarrow$$

$$E\left[\underline{u}^B(n) \cdot \underline{u}^{BH}(n)\, \underline{w}_b^B\right] = \underline{r}^* \Leftrightarrow$$

$$\underbrace{E\left[\underline{u}^B(n) \cdot \underline{u}^{BH}\right]}_{\underline{\underline{R}}^T} \cdot \underline{w}_b^B = \underline{r}^* \Leftrightarrow \underline{\underline{R}}^T \cdot \underline{w}_b^B = \underline{r}^* \Leftrightarrow$$

$$\underline{\underline{R}}^{T*} \cdot \underline{w}_b^{B*} = \underline{r} \Leftrightarrow \underline{\underline{R}}^H \cdot \underline{w}_b^{B*} = \underline{r} \overset{\underline{\underline{R}}^H = \underline{\underline{R}}}{\Longleftrightarrow}$$

$$\left.\begin{aligned} &\underline{\underline{R}} \cdot \underline{w}_b^{B*} = \underline{r} \\ \text{άρα } &\underline{\underline{R}} \cdot \underline{w}_f = \underline{r} \end{aligned}\right\} \Rightarrow \boxed{\underline{w}_b^{B*} = \underline{w}_f}$$

β. σχέση μεταξύ των ισχύων των σφαλμάτων

FPE: $P_M = E\left[|f_M(n)|^2\right] = r(0) - \underline{r}^H \underline{w}_f$

BPE: $E\left[|b_M(n)|^2\right] = r(0) - \underline{r}^{BT} \underline{w}_b = r(0) - \underline{r}^T \underline{w}_b^B =$

$$= r(0) - \underline{r}^{T*} \underline{w}_b^{B*} = r(0) - \underline{r}^H \underline{w}_f = P_M$$

Levinson - Durbin Algorithm (1)

Forward $\quad \underline{\underline{R}}_M \cdot \underline{w}_f = \underline{r}_M$

$$\underline{\underline{R}}_M = \begin{bmatrix} r(0) & r(1) & \cdots & r(M-1) \\ r^*(1) & r(0) & r(1) & r(M-2) \\ r^*(2) & & & r(M-3) \\ & & & \\ r^*(M-1) & r^*(M-2) & & r(0) \end{bmatrix}$$

$$\underline{w}_f = \begin{bmatrix} w_{f,1} \\ w_{f,2} \\ w_{f,3} \\ \vdots \\ w_{f,M} \end{bmatrix}$$

$$\underline{r}_M = \begin{bmatrix} r^*(1) \\ r^*(2) \\ \vdots \\ r^*(M) \end{bmatrix}$$

Όπου :

$$\begin{bmatrix} r(0) & r(1) & \cdots & r(N-1) \\ r^*(1) & r(0) & r(1) & r(N-2) \\ r^*(2) & & & \vdots \\ \vdots & & & r(1) \\ r^*(N-1) & \cdots & r^*(1) & r(0) \end{bmatrix} \begin{bmatrix} w_{M,1} \\ w_{M,2} \\ \vdots \\ w_{N,N-1} \\ w_{M,M} \end{bmatrix} = \begin{bmatrix} r^*(1) \\ r^*(2) \\ \vdots \\ r^*(N) \end{bmatrix} \qquad (0)$$

$\underline{\underline{R}}_{M-1}$, $\underline{r}^{B*}_{M-1}$, $\underline{\underline{L}} \cdot \underline{w}_M$, $\underline{r}_{M-1}$, $\underline{r}^{BT}_{M-1}$

όπου

$$\underline{\underline{L}} = \begin{array}{c} \\ M-1 \\ \\ \end{array} \begin{bmatrix} 1, & 0, & 0, & \cdots & 0 \\ 0, & 1, & 0, & \cdots & 0 \\ 0, & & & & 0 \\ \vdots & & & & 0 \\ 0, & 0, & \cdots & 1, & 0 \end{bmatrix}$$

$$M$$

οπότε

$$\underline{\underline{L}} \cdot \underline{w}_M = \begin{bmatrix} w_{M,1} \\ w_{N,2} \\ \vdots \\ w_{M,N-1} \end{bmatrix}$$

οπότε

$$\begin{bmatrix} \underline{\underline{R}}_{M-1} & \underline{r}^{B*}_{M-1} \\ \underline{r}^{BT}_{M-1} & r(0) \end{bmatrix} \begin{bmatrix} \underline{\underline{L}}\ \underline{w}_M \\ \\ w_{M,M} \end{bmatrix} = \begin{bmatrix} \underline{r}_{M-1} \\ \\ r^*(M) \end{bmatrix} \Rightarrow$$

$$\Rightarrow \quad \underline{\underline{R}}_{M-1} \cdot \underline{\underline{L}}\ \underline{w}_M + \underline{r}^{B*}_{M-1} w_{M,M} = \underline{r}_{M-1} \qquad (1)$$

$$\underline{r}^{BT}_{M-1} \underline{\underline{L}}\ \underline{w}_M + r(0)\, w_{M,M} = r^*(M) \qquad (2).$$

αλλά για $(M-1)^{ης}$ τάξης FLP filter ισχύει

$$\underline{\underline{R}}_{M-1} \cdot \underline{w}_{M-1} = \underline{r}_{M-1} \quad (3)$$

και για $(M-1)^{ης}$ τάξης BLP filter ισχύει

$$\underline{\underline{R}}_{M-1} \underline{w}^{B*}_{M-1} = \underline{r}^{B*}_{M-1} \quad (4)$$

εξήγηση της (4)

→ από θεωρία ξέρω για το BLP: $\underline{\underline{R}}\, \underline{w}_b = \underline{r}^{B*}$

και $\underline{w}_b^{B*} = \underline{w}_f \Rightarrow \underline{w}_b^{B} = \underline{w}_f^{*} \Rightarrow \underline{w}_b = \underline{w}_f^{B*}$ $\Bigg\} \Rightarrow$

$$\Rightarrow \quad \underline{\underline{R}}\, \underline{w}_f^{B*} = \underline{r}^{B*} \Rightarrow \underline{\underline{R}}\, \underline{w}^{B*} = \underline{r}^{B*}$$

$$(1) \xrightarrow{(3),(4)} \underline{\underline{R}}_{M-1} \cdot \underline{\underline{L}}\, \underline{w}_M + \overbrace{\underline{\underline{R}}_{M-1} \underline{w}^{B*}_{M-1}}^{(4)} w_{M,M} = \overbrace{\underline{\underline{R}}_{M-1} \cdot \underline{w}_{M-1}}^{(3)}$$

$$\xrightarrow{\underline{\underline{R}}_{M-1}^{-1} \times} \underline{\underline{L}}\, \underline{w}_M + \underline{w}^{B*}_{M-1} w_{M,M} = \underline{w}_{M-1} \Rightarrow$$

$$\underline{\underline{L}}\, \underline{w}_M = \underline{w}_{M-1} - \underline{w}^{B*}_{M-1} w_{M,M} \quad (5)$$

$$(2) \xrightarrow{(5)} \underline{r}^{BT}_{M-1} \left(\underline{w}_{M-1} - \underline{w}^{B*}_{M-1} w_{M,M} \right) + r(0)\, w_{M,M} = r^*(M) \Rightarrow$$

$$\Rightarrow \underline{r}^{BT}_{M-1} \underline{w}_{M-1} - \underline{r}^{BT}_{M-1} \underline{w}^{B*}_{M-1} w_{M,M} + r(0)\, w_{M,M} = r^*(M) \Rightarrow$$

$$\Rightarrow \quad w_{M,M} = \frac{r^*(M) - \underline{r}^{BT}_{M-1} \underline{w}_{M-1}}{r(0) - \underline{r}^{BT}_{M-1} \underline{w}^{B*}_{M-1}} \quad (6)$$

Define $k_M \triangleq -w_{M,M}$ (reflection coefficients) 4

από την (6) $\underline{w}_M = \begin{bmatrix} w_{M,1} \\ w_{M,2} \\ \vdots \\ w_{M,M} \end{bmatrix} = \begin{bmatrix} \underline{L}\,\underline{w}_M \\ \\ w_{M,M} \end{bmatrix} \overset{(5)}{=} \begin{bmatrix} \underline{w}_{M-1} - \underline{w}^{B*}_{M-1} w_{M,M} \\ \\ w_{M,M} \end{bmatrix} =$

$$= \begin{bmatrix} \underline{w}_{M-1} \\ \\ 0 \end{bmatrix} - w_{M,M} \begin{bmatrix} \underline{w}^{B*}_{M-1} \\ \\ -1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \underline{w}_M = \begin{bmatrix} \underline{w}_{M-1} \\ \\ 0 \end{bmatrix} + k_M \begin{bmatrix} \underline{w}^{B*}_{M-1} \\ \\ -1 \end{bmatrix} \quad (7)$$

Ορίζουμε $\underline{a}_M = \begin{bmatrix} 1 \\ -\underline{w}_M \end{bmatrix}$ και $\underline{a}_{M-1} = \begin{bmatrix} 1 \\ -\underline{w}_{M-1} \end{bmatrix}$

οπότε $\underline{a}_M = \begin{bmatrix} 1 \\ -\underline{w}_M \end{bmatrix} \overset{(7)}{=} \begin{bmatrix} 1 \\ \begin{bmatrix} -\underline{w}_{M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} -\underline{w}^{B*}_{M-1} \\ +1 \end{bmatrix} \end{bmatrix} =$

$$= \begin{bmatrix} 1 + 0 \\ \begin{bmatrix} -\underline{w}_{M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} \underline{w}^{B*}_{M-1} \\ -1 \end{bmatrix} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -\underline{w}_{M-1} \\ 0 \end{bmatrix}}^{\underline{a}_{M-1}} + k_M \begin{bmatrix} 0 \\ -\underline{w}^{B*}_{M-1} \\ +1 \end{bmatrix} =$$

$$= \boxed{\begin{bmatrix} \underline{a}_{M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} 0 \\ \underline{a}^{B*}_{M-1} \end{bmatrix} = \underline{a}_M} \quad (8)$$

In scalar form

από (8)

$$\underline{a}_M = \begin{bmatrix} 1 \\ -w_{M,1} \\ -w_{M,2} \\ -w_{M,3} \\ \vdots \\ -w_{M,M} \end{bmatrix} = \begin{bmatrix} 1 \\ -w_{M-1,1} \\ -w_{M-1,2} \\ -w_{M-1,3} \\ \vdots \\ -w_{M-1,M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} 0 \\ -w^*_{M-1,M-1} \\ -w^*_{M-1,M-2} \\ -w^*_{M-1,M-3} \\ \vdots \\ -w^*_{M-1,1} \\ 1 \end{bmatrix} \Rightarrow$$

$$\underline{a}_M = \begin{bmatrix} 1 \\ a_{M,1} \\ a_{M,2} \\ \vdots \\ a_{M,M} \end{bmatrix} = \begin{bmatrix} 1 \\ a_{M-1,1} \\ a_{M-1,2} \\ \vdots \\ a_{M-1,M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} 0 \\ a^*_{M-1,M-1} \\ a^*_{M-1,M-2} \\ \vdots \\ a^*_{M-1,1} \\ 1 \end{bmatrix} \Rightarrow$$

$$\boxed{a_{M,k} = a_{M-1,k} + k_M a^*_{M-1,M-k}, \quad k = 0,1,2,\ldots,M-1 \qquad \text{με } a_{M,0} = 1 \text{ και } a_{M-1,0} = 1} \quad (9)$$

Από την (6) ορίζω

τον αριθμητή ως: $\Delta_{M-1} \triangleq r^*(M) - \underline{r}^{BT}_{M-1}\underline{w}_{M-1} =$

$= r^*(M)\cdot 1 - \underline{r}^{BT}_{M-1}\underline{w}_{M-1} \overset{\underline{a}_{M-1} = \left[\begin{smallmatrix}1 \\ -\underline{w}_{M-1}\end{smallmatrix}\right]}{=} \underline{r}^{BT}_M \cdot \underline{a}_{M-1} \overset{*}{=}$ $\quad$ * because R is hermitian $r(-k) = r^*(k)$

$= \sum_{l=0}^{M-1} a_{M-1,l}\, r(l-M) = E\left[b_{M-1}(n-1)\, f^*_{M-1}(n)\right]$ ← πιθανότατα ισοδύναμα προς απόδειξη στην 3η σειρά ασκήσεων

όπου $b_M(n) = u(n-M) - \sum_{k=1}^{M} w^*_{b,k}\, u(n-k+1)$ backward prediction error

και $f_M(n) = u(n) - \sum_{k=1}^{M} w^*_{f,k}\, u(n-k)$ forward prediction error

και ο παρονομαστής της (6) είναι απλά η ισχύς του ελάχιστου σφάλματος LBP (6)
τάξης $M-1$: $r(0)-\underline{r}^{BT}_{M-1}\cdot\underline{w}^{B*}_{M-1} = P_{M-1}$

[από σχέση (6.10) Haykin : $P_M = r(0)-\underline{r}^{H}_{M}\underline{w}_{f,M} = r(0)-\underline{r}^{T}_{M}\underline{w}^{*}_{f,M} =$
$= r(0)-\underline{r}^{TB}_{M}\cdot\underline{w}^{B*}_{f,M}$]

Οπότε η σχέση (6) γίνεται

$$k_M \triangleq -w_{M,M} = \frac{-\Delta_{M-1}}{P_{M-1}} = \frac{-\sum_{l=0}^{M-1} a_{M-1,l}\, r(l-M)}{P_{M-1}} = \frac{-E[b_{M-1}^{(n-1)} f^{*}_{M-1}(n)]}{E[|f_{M-1}(n)|^2]}$$

$$\Rightarrow \boxed{k_M = \frac{-\sum_{l=0}^{M-1} a_{M-1,l}\, r(l-M)}{P_{M-1}}} \quad (10) \quad \text{[αναδρομική σχέση]}$$

και από 6.10 Haykin : $P_M = r(0)-\underline{r}^{H}_{M}\cdot\underline{w}_{M} \overset{(7)}{\Rightarrow}$

$$P_M = r(0)-\underline{r}^{H}_{M}\left(\begin{bmatrix}\underline{w}_{M-1}\\0\end{bmatrix}+k_M\begin{bmatrix}\underline{w}^{B*}_{M-1}\\-1\end{bmatrix}\right) =$$

$$\underbrace{r(0)-\underline{r}^{H}_{M-1}\underline{w}_{M-1}}_{P_{M-1}} - k_M\,\underline{r}^{H}_{M-1}\underline{w}^{B*}_{M-1} + k_M\, r(M) \Rightarrow$$

$$P_M = P_{M-1} + k_M\left(r(M)-\underline{r}^{H}_{M-1}\underline{w}^{B*}_{M-1}\right) = P_{M-1} + k_M\left(r(M)-\underline{r}^{BH}_{M-1}\underline{w}^{*}_{M-1}\right) =$$

$$P_{M-1} + k_M\Big(\underbrace{r^{*}(M)-\underline{r}^{BT}_{M-1}\underline{w}_{M-1}}_{\Delta_{M-1}}\Big)^{*} \Rightarrow P_M = P_{M-1}+k_M\,\Delta^{*}_{M-1} \overset{(10)}{\Longrightarrow}$$

$$P_M = P_{M-1}+k_M\left(-k_M P_{M-1}\right)^{*} \Rightarrow P_M = P_{M-1}-|k_M|^2 P_{M-1} \Rightarrow$$

$$\Rightarrow \boxed{P_M = (1-|k_M|^2)\,P_{M-1} = P_0\prod_{l=1}^{M}(1-|k_l|^2)} \quad (11)$$

αναδρομική σχέση

Οπότε ο αλγόριθμος Levinson-Durbin

- $P_0 = r(0)$

- $m = 1, 2, 3, 4, \ldots, M$

$$k_m = a_{m,m} = \frac{-\Delta_{m-1}}{P_{m-1}} = \frac{-\sum_{l=0}^{m-1} a_{m-1,l}\, r(l-m)}{P_{m-1}}$$

$$\left[\Delta_0 = \sum_{l=0}^{0} a_{0,0}\, r(0-1) = a_{0,0}\, r(-1) = r(-1) = r^*(1)\right]$$

$k = 0, 1, 2, \ldots, m-1$

$$a_{m,k} = a_{m-1,k} + k_m a^*_{m-1,m-k} \quad (a_{l,m} = 0 \;;\; m > l)$$

(σχέση (a))

end

$$P_m = (1 - |k_m|^2) P_{m-1}$$

end