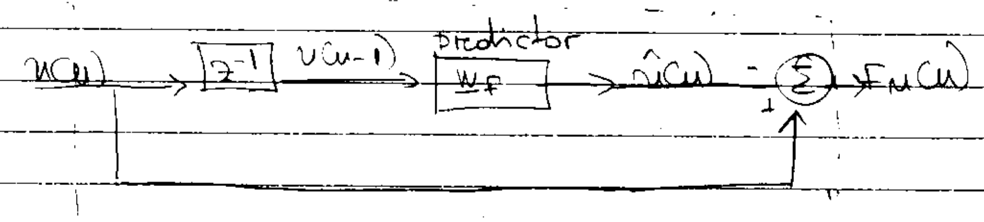
22/03/2010

Στατιστική Επεξεργασία Σήματος

12th lecture.

FLP (Forward linear prediction)



↳ predictor error filter

Problem Statement: Given the M signal values: $[u(n-1), u(n-2), \ldots u(n-M)]^T$
$= \underline{u}_M(n-1)$

Find the optimum weights: $[w_{f_1}, \ldots, w_{f_M}]^T = \underline{w}_f$
such that:

if $\left(\hat{u}(n) \mid \underline{u}_M(n-1)\right) = \sum_{k=1}^{M} w_{f_k}^* u(n-k) = \underline{w}_f^H \cdot \underline{u}_M(n-1)$ then

the error $f_M(n) = u(n) - \hat{u}(n \mid \underline{u}_M(n-1))$ is minimized in the MSE sense, ie,

$$\min_{\underline{w}_f} \underbrace{E\left[|f_M(n)|^2\right]}_{P_M} \rightarrow \text{forward prediction error power}$$

Solution: $\underline{\underline{R}}_M \cdot \underline{w}_f = \underline{r}_M = \begin{bmatrix} r^*(1) \\ \vdots \\ r^*(M) \end{bmatrix}$

$P_M = r(0) - \underline{r}_M^H \cdot \underline{w}_f$

Augmented WH Eqs:

$$\underbrace{\begin{bmatrix} r(0) & \underline{r}_M^H \\ \underline{r}_M & \underline{\underline{R}}_M \end{bmatrix}}_{\underline{\underline{R}}_{M+1}} \cdot \underbrace{\begin{bmatrix} 1 \\ -\underline{w}_f \end{bmatrix}}_{\underline{a}_M} = \begin{bmatrix} P_M \\ \underline{0} \end{bmatrix} = \begin{bmatrix} P_M \\ \underline{0} \end{bmatrix}$$

$$f_M(n) = \begin{bmatrix} 1 \\ -\underline{w}_F \end{bmatrix}^H \underline{u}_{M+1}(n) = \underline{a}_M^H \cdot \underline{u}_{M+1}(n)$$

I/O of the FLPEF

BLP (backward linear predictor)

Given the M signal values: $[u(n), \dots, u(n-M+1)] = \underline{u}_M(n)$ find the opt. weights $[w_{b,1}, \dots, w_{b,M}]^T = \underline{w}_b$ s.t. $b_M(n) = u(n-M) - \underline{w}_b^H \underline{u}_M(n)$ is minimized in the MSE sense

$$\min_{\underline{w}_b} E[|b_M(n)|^2]$$

Solution: $\boxed{\underline{\underline{R}}_M \cdot \underline{w}_b = \underline{r}_M^{B*}}$

$$P(M) = r(0) - \underline{r}_M^{BT} \cdot \underline{w}_b$$

μηδενικό διάνυσμα

Augmented WH Eqs: $\underline{\underline{R}}_{M+1}\, \underline{c}_M = \begin{bmatrix} \underline{0} \\ P_M \end{bmatrix}$

$\underline{b}_M(n) = \underline{c}_M^H \; \underline{u}_{M+1}(n)$: I/O BLPEF

We showed that: $\underline{w}_F = \underline{w}_B^{B*}$

note: Αν έχω τετραγωνικό πίνακα M×M το computational complexity της αντιστροφής του πίνακα αυτού είναι M^3 με gauss-elimination

LEVINSON-DURBIN Algorithm

Basic Idea: Solve WH Eqs for optimum weights for the first 1st, 2nd, 3rd, ... Mth order recursively ⇒ reduce the computational complexity induced by Gaussian elimination (matrix inversion) method.

Derivation: Mth order FP: $\underline{\underline{R}}_M \cdot \underline{w}_M = \underline{r}_M \Rightarrow$

$\longrightarrow$

$\underline{\underline{R}}_{M-1}$ $\quad \underline{r}^{B*}_{M-1} \quad \underline{\underline{L}}\cdot\underline{w}_M \quad \underline{r}_{M-1}$

$$\Rightarrow \left[\begin{array}{cccc|c} r(0) & r(1) & \cdots & & r(M-1)\\ r^*(1) & r(0) & & & r(M-2)\\ r^*(2) & r^*(1) & r(0) & & \vdots\\ \vdots & & \underline{\underline{R}}_{M-1} & & \vdots\\ \hline r^*(M-1) & r^*(M-2) & \cdots & & r(0)\end{array}\right] \begin{bmatrix} w_{M,1}\\ w_{M,2}\\ \vdots\\ \vdots\\ w_{M,M}\end{bmatrix} = \begin{bmatrix} r^*(1)\\ r^*(2)\\ r^*(3)\\ \vdots\\ r^*(M)\end{bmatrix}$$

$\underline{r}^{BT}_{M-1}$

note : ξέρουμε [illegible] και $\underline{\underline{R}}_{N-1}\,\underline{w}_N = \underline{r}_{N-1}$

Let $$\underline{\underline{L}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & \\ \vdots & & & & & \\ 0 & 0 & \cdots & & 1 & 0 \end{bmatrix} \quad \begin{array}{l}\uparrow\\ M-1\\ \downarrow\end{array}$$

$\longleftarrow M \longrightarrow$

$\underline{\underline{L}}$ $(M-1)\times M$

then, $$\left[\begin{array}{c|c} \underline{\underline{R}}_{M-1} & \underline{r}^{B*}_{M-1}\\ \hline \underline{r}^{BT}_{M-1} & r(0)\end{array}\right] \begin{bmatrix} \underline{\underline{L}}\cdot\underline{w}_M\\ \\ w_{M,M}\end{bmatrix} = \begin{bmatrix} \underline{r}_{M-1}\\ \\ r^*(M)\end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \quad \underline{\underline{R}}_{M-1}\cdot\underline{\underline{L}}\cdot\underline{w}_M + \underline{r}^{B*}_{M-1}\cdot w_{M,M} = \underline{r}_{M-1} \qquad (1)$$

$$\underline{r}^{BT}_{M-1}\cdot\underline{\underline{L}}\cdot\underline{w}_M + r(0)\,w_{M,M} = r^*(M) \qquad (2)$$

From $(M-1)^{\underline{th}}$ order FP : $\underline{\underline{R}}_{M-1}\cdot\underline{w}_{M-1} = \underline{r}_{M-1}$ (3)

From $(M-1)^{\underline{th}}$ order BP : $\underline{\underline{R}}_{M-1}\cdot\underline{w}^{B*}_{M-1} = \underline{r}^{B*}_{M-1}$ (4)

From (1), using (3) & (4), we get :

$$\underline{\underline{R}}_{M-1}\cdot\underline{\underline{L}}\cdot\underline{w}_M + \underline{\underline{R}}_{M-1}\cdot\underline{w}^{B*}_{M-1}\cdot w_{M,M} = \underline{\underline{R}}_{M-1}\cdot\underline{w}_{M-1} \Longrightarrow$$

note : θεωρούμε ότι υπάρχει ο $\underline{\underline{R}}^{-1}_{M-1}$

$$\Rightarrow \underline{\underline{L}}\,\underline{w}_M + \underline{w}_{M-1}^{B*} \cdot w_{M,M} = \underline{w}_{M-1} \quad \Rightarrow$$

$$\Rightarrow \boxed{\underline{\underline{L}} \cdot \underline{w}_M = \underline{w}_{M-1} - \underline{w}_{M-1}^{B*}\, w_{M,M}} \qquad (5)$$

(⑤ our ②)

Then, using (5) & (2), to get:

$$\underline{r}_{M-1}^{BT} \left(\underline{w}_{M-1} - \underline{w}_{M-1}^{B*} \cdot w_{M,M}\right) + r(0)\, w_{M,M} = r^*(M) \quad \Rightarrow$$

$$\boxed{w_{M,M} = \frac{r^*(M) - \underline{r}_{M-1}^{BT} \cdot \underline{w}_{M-1}}{r(0) - \underline{r}_{M-1}^{BT} \cdot \underline{w}_{M-1}^{B*}}} \qquad (6)$$

Define $k_M \triangleq -w_{M,M}$. From (5) & (6)

$$\underline{w}_M = \begin{bmatrix} \underline{\underline{L}} \cdot \underline{w}_M \\ w_{M,M} \end{bmatrix} = \begin{bmatrix} \underline{w}_{M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} \underline{w}_{M-1}^{*} \\ -1 \end{bmatrix} \qquad (7)$$

$$\underline{\alpha}_M = \begin{bmatrix} 1 \\ -\underline{w}_M \end{bmatrix}, \quad \underline{\alpha}_{M-1} = \begin{bmatrix} 1 \\ -\underline{w}_{M-1} \end{bmatrix}$$

$$(7) \Rightarrow \begin{bmatrix} 1 \\ -\underline{w}_M \end{bmatrix} = \begin{bmatrix} 1 \\ -\underline{w}_{M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} 0 \\ -\underline{w}_{M-1}^{*} \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \boxed{\underline{\alpha}_M = \begin{bmatrix} \underline{\alpha}_{M-1} \\ 0 \end{bmatrix} + k_M \begin{bmatrix} 0 \\ \underline{\alpha}_{M-1}^{B*} \end{bmatrix} \qquad (8)}$$

In scalar form:

$$\boxed{\begin{aligned} &\alpha_{M,k} = \alpha_{M-1,k} + k_M\, \alpha^*_{M-1,M-k}, \quad k = 0, 1, \dots, M-1 \\ &\alpha_{M,0} = 1 \\ &\alpha_{M-1,0} = 1 \end{aligned}} \qquad (9)$$

Define : $\Delta_{m-1} \overset{\Delta}{=} r^{*}(m) - \underline{r}_{m-1}^{BT} \cdot \underline{w}_{m-1} = \underline{r}_{m}^{BT} \cdot \underline{a}_{m-1} =$

$$= \sum_{l=0}^{m-1} a_{m-1,l}\, r(l-m) \underset{HW4}{\equiv} E\left[b_{m-1}(n-1) \cdot f_{m-1}^{*}(n)\right]$$

αυτό θα δείξω σαν ασκη

Also, $r(0) - \underline{r}_{m-1}^{BT} \cdot \underline{w}_{m-1}^{B*} = P_{m-1}$

Hence, from (6) :

$$k_m \overset{\Delta}{=} -w_{m,m} = -\frac{\Delta_{m-1}}{P_{m-1}} = -\frac{\sum_{l=0}^{m-1} a_{m-1,l}\, r(l-m)}{P_{m-1}} =$$

$$\boxed{= -\frac{E\left[b_{m-1}(n-1) \cdot f_{m-1}^{*}(n)\right]}{E\left[|f_{m-1}(n)|^2\right]} \quad (10)}$$

$$P_m = r(0) - \underline{r}_m^{H} \cdot \underline{w}_m \underset{(7)}{=} r(0) - \underline{r}_m^{H}\left(\begin{bmatrix}\underline{w}_{m-1} \\ 0\end{bmatrix} + k_m \begin{bmatrix}\underline{w}_{m-1}^{B*} \\ -1\end{bmatrix}\right) =$$

$$= r(0) - \underline{r}_{m-1}^{H} \cdot \underline{w}_{m-1} - k_m\left(\underline{r}_{m-1}^{H} \cdot \underline{w}_{m-1}^{B*} - r(m) \cdot 1\right) =$$

$$= P_{m-1} + k_m \underbrace{\left(r^{*}(m) - \underline{r}_{m-1}^{BT} \cdot \underline{w}_{m-1}\right)}_{\Delta_{m-1}}{}^{*} \Rightarrow$$

$$\Rightarrow P_m = P_{m-1} + k_m \Delta_{m-1}^{*} = P_{m-1} - |k_m|^2 P_{m-1} \Rightarrow$$

$$P_m = \left(1 - |k_m|^2\right) P_{m-1} \left(= \underset{\overset{|||}{r(0)}}{P_0} \prod_{l=1}^{m} \left(1 - |k_l|^2\right)\right)$$

Levinson - Durbin

$P_0 = r(0)$

$m = 1, 2, 3, \ldots, M$

$$k_m = a_{m,m} = \frac{-\Delta_{m-1}}{P_{m-1}} = \frac{-\sum_{l=0}^{m-1} a_{m-1,l}\, r(l-m)}{P_{m-1}}$$

$(\Delta_0 = r^{*}(1))$

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    $k = 0, 1, \ldots, m-1$
        $\alpha_{m,k} = \alpha_{m-1,k} + k_m \alpha^*_{m-1,m-k}$
    end
    $P_m = P_{m-1}(1 - |k_m|^2)$          $(\alpha_{l,m} = 0 \; m>l)$.
end
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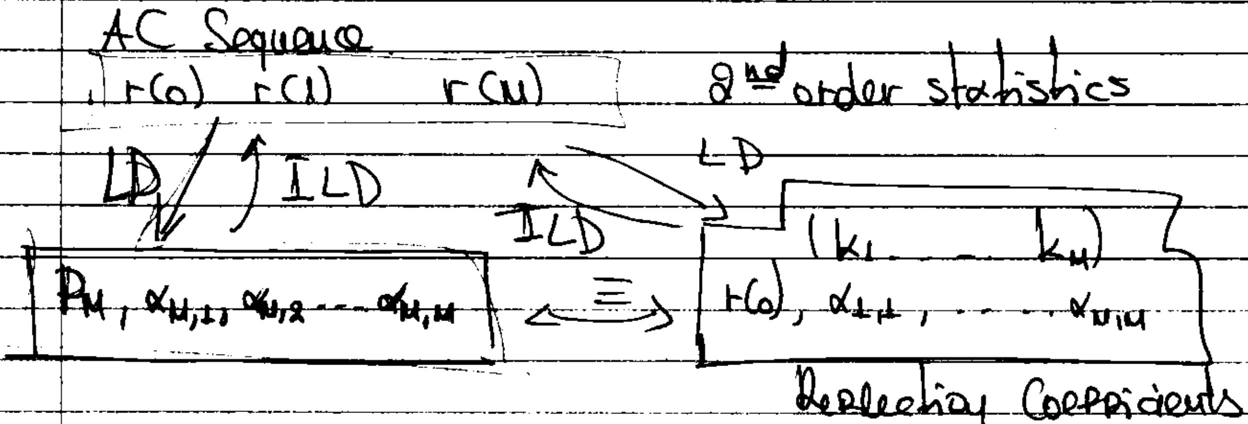

η υπολογιστική πολυπλοκότητα του αλγορίθμου Levinson-Durbin είναι $O(M^2)$.

note: ILD (inverse levinson durbin)
LD (levinson-durbin)