17/03/2010

Στατιστική Επεξεργασία Σήματος

11th Lecture

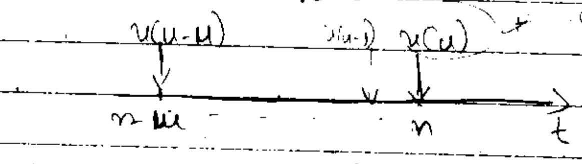
Linear Prediction

Έχω στοχ. διαδικασία στάσιμη

Goal: predict a future value of stationary s.p. given a set of past values of the process

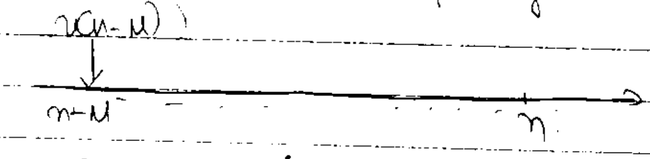
In linear prediction, we express the predicted value as a linear combination of past values

FORWARD: (FLP)



Predict $u(n)$ from $\{u(n-1), u(n-2), \dots, u(n-M)\}$

BACKWARD (BLP)



Predict $u(n-M)$ from $\{u(n-M+1), u(n-M+2), \dots u(n)\}$

- Design the optimal FLP & BLP filters.
  - Levinson-Durbin Algorithm
  - Lattice filters Δικτυωτά φίλτρα

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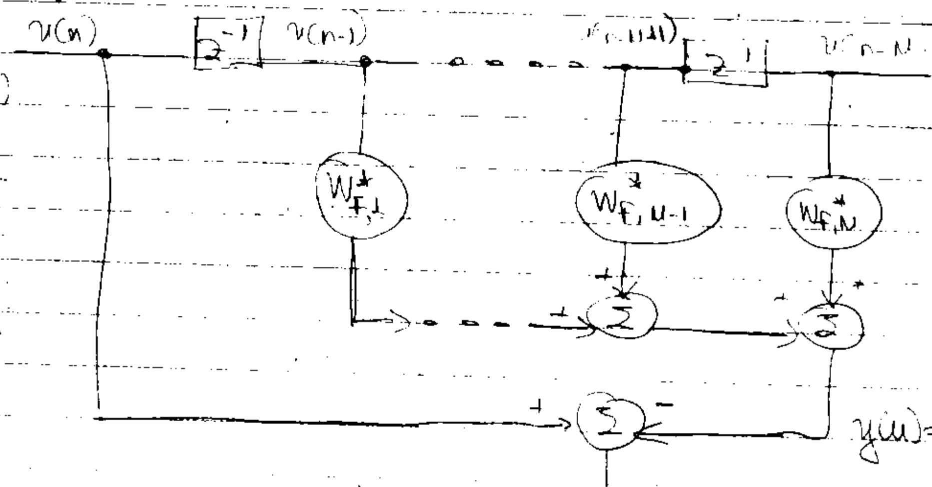
FLP

αναγκαστικά διαθέσιμα τα $u(n), \dots, u(n-M)$ για να βρω το future ... forward prediction filter



$y(n) = \hat{u}(n|\underline{u}(n))$

$f_M(n) = u(n) - \hat{u}(n|\underline{u}(n))$

forward prediction error τάξης M

(1)

- $y(n) = \hat{u}(n \mid \underline{u}(n)) = \sum_{k=1}^{M} w_{f,k}^{*}\, u(n-k)$ = Filter output

- $f_M(n) = u(n) - \hat{u}(n \mid \underline{u}(n))$ = FLP error

ισχύς του σφάλματος

- $P_M = E\left[\,|f_M(n)|^2\,\right]$ = MS prediction error

Κριτήριο: να ελαχιστοποιήσω την ισχύ του σφάλματος

$\min_{\underline{w}_f} P_M$ where $\underline{w}_f = [w_{f,1}, w_{f,2}, \ldots, w_{f,M}]^T$

$d(n) = u(n)$

$\underline{u}(n-1) = [u(n-1), \ldots, u(n-M)]^T$

Correlation Matrix

$$\underline{\underline{R}} = E[\underline{u}(n-1)\underline{u}^H(n-1)] \Rightarrow$$

$$\Rightarrow \underline{\underline{R}} = E\left\{ \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(n-M) \end{bmatrix} \begin{bmatrix} u^*(n-1) & u^*(n-2) & \cdots & u^*(n-M) \end{bmatrix} \right\} =$$

$$= \begin{bmatrix} r(0) & r(1) & \cdots & r(M-1) \\ r^*(1) & r(0) & & r(M-2) \\ r^*(2) & & \ddots & \vdots \\ \vdots & & & \vdots \\ r^*(M-1) & r^*(M-2) & & r(0) \end{bmatrix}$$

Cross-correlation Vector

$$\underline{p} = E[\underline{u}(n-1) \cdot u^*(n)] = E\left[\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(n-M) \end{bmatrix} u^*(n) \right] = \begin{bmatrix} r^*(1) \\ r^*(2) \\ \vdots \\ r^*(M) \end{bmatrix} \triangleq \underline{r}$$

Hence the optimal solution of FLP filter

$$\boxed{\underline{\underline{R}} \cdot \underline{w}_f = \underline{r}} \quad (1)$$

(Αυτό είναι το σύστημα of Yule-Walker equations που ισχύει για τις AR διαδικασίες)

The resulting Error

$$P_M = \sigma_d^2 - \underline{p}^H \underline{w}_f = r(0) - \underline{r}^H \cdot \underline{w}_f = r(0) - \underline{r}^H \cdot \underline{\underline{R}}^{-1} \cdot \underline{r} \Rightarrow$$

$$\Rightarrow \boxed{P_M = r(0) - \underline{r}^H \cdot \underline{\underline{R}}^{-1} \cdot \underline{r}} \quad (2)$$

Combine Eqs (1) & (2) to get the augmented WH Eqs for FP:

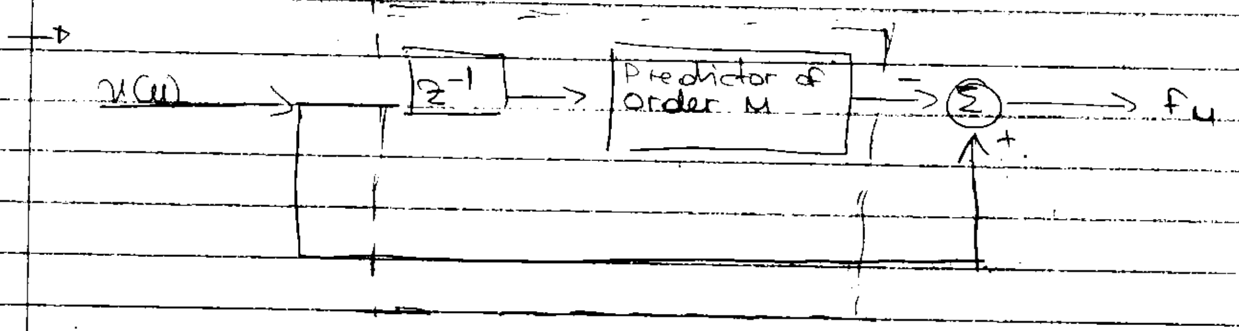
$$\underbrace{\begin{bmatrix} r(0) & \underline{r}^H \\ \underline{r} & \underline{\underline{R}}_M \end{bmatrix}}_{\underline{\underline{R}}_{M+1}} \underbrace{\begin{bmatrix} 1 \\ -\underline{w}_f \end{bmatrix}}_{\underline{a}_M} = \begin{bmatrix} P_M \\ \underline{0} \end{bmatrix}$$

→ corr. matrix of $\underline{u}_{M+1}(n) = [u(n), \ldots, u(n-M)]$

$$\underline{a}_M = \begin{cases} 1 & k=0 \\ -w_{f,k} & k = 1 \ldots M \end{cases}$$ is the forward prediction error filter coefficients

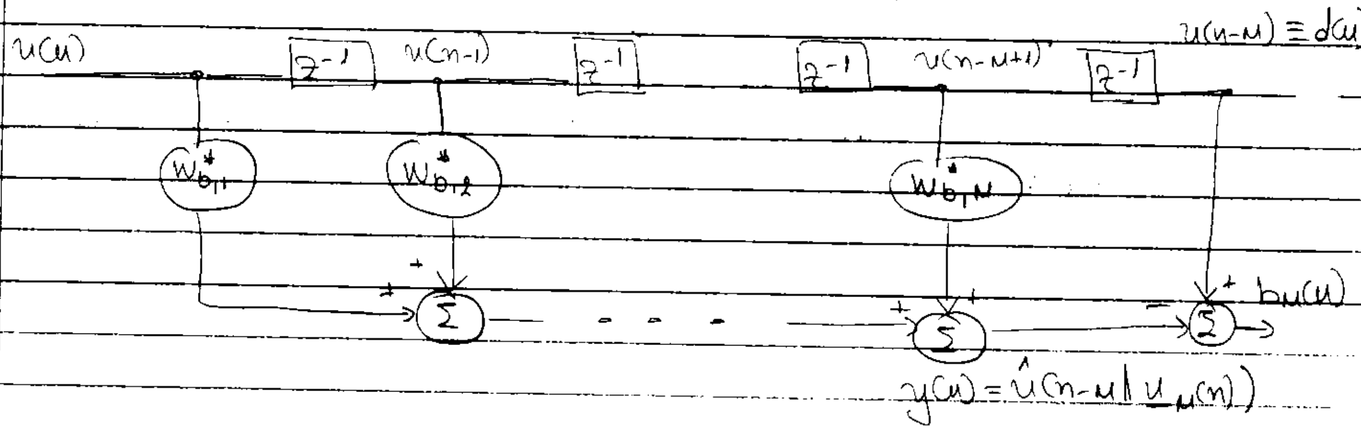
$$f_M(n) = u(n) - \underline{w}_f^H \cdot \underline{u}(n-1) = \underline{a}_M^H \cdot \underline{u}_{M+1}(n)$$ [illegible]

LPF & AR processes: The WH Eqs are identical to the Yule-Walker equations, for an M-th order AR process. Hence, for an AR(M) process of known order M, when a FP is optimized in the MS sense, its tap weights take on the same values as the parameters of the process



$\underline{a}_M$

BLP (backward linear prediction)



σφάλμα $\quad b_M(n) = u(n-M) - \hat{u}(n-M \mid \underline{u}_M(n)) = u(n-M) - \sum_{k=1}^{M} w_{b,k}^{*}\, u(n-k+1)$

Τι θέλω; να ελαχιστοποιήσω την ισχύ του σφάλματος εκτίμησης $b_M(n)$
άρα

$$\min_{\underline{w}_b} P_M = E\left[\,|b_M(n)|^2\,\right]$$

- $\underline{w}_b = [w_{b,1}, \ldots\ldots, w_{b,M}]^T$
- $\underline{u}(n) = [u(n), u(n-1), \ldots, u(n-M+1)]^T$
- $d(n) = u(n-M)$
- $\underline{\underline{R}} = E[\underline{u}(n)\underline{u}^H(n)]$
- $\underline{p} = E[\underline{u}(n) \cdot d^*(n)] = E\left[\begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(n-M+1) \end{bmatrix} u^*(n-M)\right] = \begin{bmatrix} r(M) \\ r(M-1) \\ \vdots \\ r(1) \end{bmatrix} = \underline{r}^{B*}$

οπότε WH Eqs : $\boxed{\underline{\underline{R}} \cdot \underline{w}_b = \underline{r}^{B*}}$

BLP error power : $P_M = r(0) - \underline{r}^{BT} \cdot \underline{w}_b$

Augmented WH Eqs

$$\underbrace{\begin{bmatrix} \underline{\underline{R}} & \underline{r}^{B*} \\ \underline{r}^{BT} & r(0) \end{bmatrix}}_{\underline{\underline{R}}_{M+1}} \underbrace{\begin{bmatrix} -\underline{w}_B \\ 1 \end{bmatrix}}_{\underline{c}_M} = \begin{bmatrix} \underline{0} \\ P_M \end{bmatrix}$$

where

$$c_{M,k} = \begin{cases} -w_{b,k+1} & k=0,1,\dots,M-1 \\ 1 & k=M \end{cases}$$

and $b_M(n) = \sum_{k=0}^{M} c_{M,k}^{*} u(n-k)$

Relation between FLP & BLP

FLP: $\underline{\underline{R}} \cdot \underline{w}_f = \underline{r}$

BLP: $\underline{\underline{R}} \cdot \underline{w}_b = \underline{r}^{B*} \iff E[\underline{u}(n) \cdot \underline{u}^{H}(n) \cdot \underline{w}_b] = \underline{r}^{B*} \iff$

$\iff E[\underline{u}(n) \cdot \underline{u}^{BH}(n) \cdot \underline{w}_b^{B}] = \underline{r}^{B*} \iff \left(E[\underline{u}(n) \cdot \underline{u}^{BH}(n) \cdot \underline{w}_b^{B}]\right)^{B} = (\underline{r}^{B*})^{B} \iff$

← εδώ δε να παίζει κάτι για σίγουρο

$\iff E[\underline{u}^{B}(n) \cdot \underline{u}^{BH}(n) \cdot \underline{w}_b^{B}] = \underline{r}^{*} \iff \underbrace{E[\underline{u}^{B}(n)\, \underline{u}^{BH}(n)]}_{\underline{\underline{R}}^{T}} \cdot \underline{w}_b^{B} = \underline{r}^{*} \iff$

$\iff \underline{\underline{R}}^{T} \cdot \underline{w}_b^{B} = \underline{r}^{*} \iff \underline{\underline{R}}^{T*} \underline{w}_b^{B*} = \underline{r} \iff$

$$\left.\begin{aligned} &\iff \underline{\underline{R}}^{H} \cdot \underline{w}_b^{B*} = \underline{r} \iff \underline{\underline{R}} \cdot \underline{w}_b^{B*} = \underline{r} \\ &\text{αλλά} \qquad \underline{\underline{R}} \cdot \underline{w}_f = \underline{r} \end{aligned}\right\} \Rightarrow \underline{w}_f = \underline{w}_b^{B*}$$

σχέση μεταξύ των ισχύων σφαλμάτων

FPE $\quad P_M = r(0) - \underline{r}^{H} \cdot \underline{w}_f$

BPE $\quad P_M = r(0) - \underline{r}^{BT} \underline{w}_b = r(0) - \underline{r}^{T} \underline{w}_b^{B} =$

$$= r(0) - \underline{r}^{T*} \cdot \underline{w}_b^{B*} = r(0) - \underline{r}^{H} \cdot \underline{w}_f$$

(εισάγουμε συζυγές χωρίς να αλλάξει και γιατί επίσης είναι ίσος το $\underline{r}^T \underline{w}_b^B$ είναι πραγματικό οπότε το συζυγές είναι αυτό)