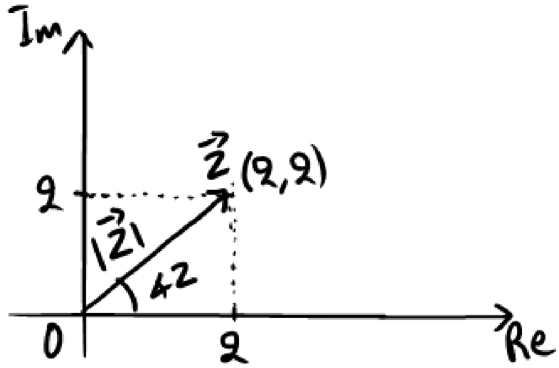
ΗΥ215
Λύσεις 1ης σειράς ασκήσεων

1. **Άλγεβρα μιγαδικών αριθμών Ι**

   (α′) $|z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.
   $\angle z = \tan^{-1}\dfrac{Im\{z\}}{Re\{z\}} = \tan^{-1}\dfrac{2}{2} = \tan^{-1} 1 = \dfrac{\pi}{4}$, γιατί $Re\{z\} > 0, Im\{z\} > 0$.



Σχήμα 1: Άσκηση 1α

   (β′) $|z| = |u| = |v| = |w| = 2\sqrt{2}$. Όμως $\angle z = \frac{\pi}{4}, \angle u = \tan^{-1}\frac{+2}{-2} = \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$, γιατί ο $\overrightarrow{u}$ ανήκει στο 2ο τεταρτημόριο ($[\frac{\pi}{2}, \pi)$). Όμοια, $\angle u = \tan^{-1}\frac{-2}{-2} = \tan^{-1} 1 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$, και $\angle w = \tan^{-1}\frac{-2}{2} = \tan^{-1}(-1) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$(ή$-\frac{\pi}{4}$). Πάντα εξετάζουμε που βρίσκεται ο μιγαδικός πριν υπολογίσουμε φάση!

   Αλγεβρικά, $w + z + v + u = 2 + 2j + 2 - 2j - 2 + 2j - 2 - 2j = 0$, άρα το μηδενικό διάνυσμα είναι το άθροισμα όλων των διανυσμάτων.

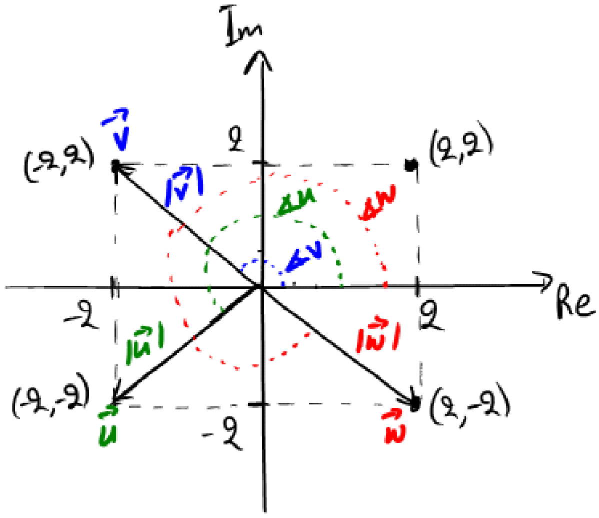
   Ίδιο μέτρο, αντίθετη κατεύθυνση, άρα το άθροισμα είναι $\overrightarrow{0}$.

   (γ′)

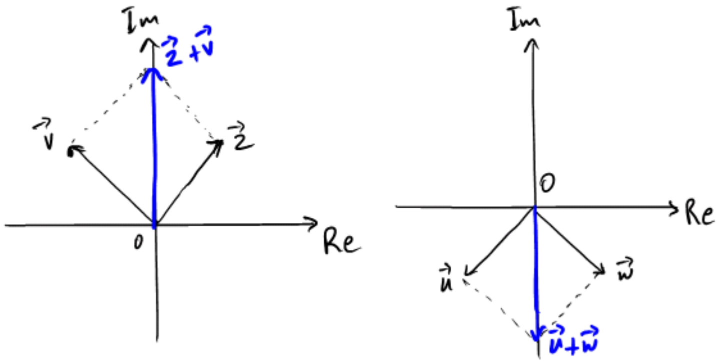
$$\frac{z}{w} = \frac{2+2j}{2-2j} = \frac{(2+2j)(2+2j)}{|2+2j|^2} = \frac{4+4j+4j-4}{2^2+2^2} = \frac{8j}{8} = j,\ |\frac{z}{w}| = |j| = 1,\ \angle\frac{z}{w} = \frac{\pi}{2}$$

$$\frac{w}{v} = \frac{2-2j}{-2+2j} = \frac{(2-2j)(-2-2j)}{(-2)^2+2^2} = \frac{-4-4j+4j-4}{4+4} = \frac{-8}{8} = -1\,|\frac{w}{v}| = |-1| = 1,\ \angle\frac{w}{v} = \pi$$

$$\frac{u}{z} = \frac{-2-2j}{2+2j} = \frac{(-2-2j)(2-2j)}{2^2+2^2} = \frac{-4+4j-4j-4}{4+4} = \frac{-8}{8} = -1,\ |\frac{u}{z}| = 1,\ \angle\frac{u}{z} = \pi$$



Σχήμα 2: Άσκηση 1β-1



Σχήμα 3: Άσκηση 1β-2

2. **Άλγεβρα μιγαδικών αριθμών ΙΙ**

(α′) $\ln(w) = \ln(e^z) = z \ln e = 1 + 1j$

(β′) $w = e^z = e^{1+1j} = ee^j \Leftrightarrow w = e(\cos(1) + j\sin(1)) \Leftrightarrow Re\{w\} = e\cos(1),\;\; Im\{w\} = e\sin(1)$

(γ′) $w + w^* = e^z = (e^z)^* = e^z + e^{z^*} = ee^j + ee^{-j} = e(e^j + e^{-j}) = 2e\cos(1)$

(δ′) $|w| = |ee^j| = |e|1 = e$

$\angle w = 1$, γιατί $w = |w|e^{j\angle w} = ee^{j1}$

Αλλιώς,

$$\angle w = \tan^{-1}\frac{Im\{w\}}{Re\{w\}} = \tan^{-1}\frac{e\sin(1)}{e\cos(1)} = \tan^{-1}(\tan(1)) = 1$$

(ε′) $|\ln(w)|^2 = |1 + 1j|^2 = 1^2 + 1^2 = 2$

(ϛ') $w + w^* = 2e\cos(1)$, από (γ) ερώτημα. Άρα, $\cos(1) = \frac{w+w^*}{2e}$

3. **Άλγεβρα μιγαδικών αριθμών III**

   (α') i. $zz^* = (x+jy)(x-jy) = x^2 - xyj + xyj + y^2 = x^2 + y^2$.

   ii. $\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}$

   (β') i. $zz^* = |z|e^{j\theta}|z^*|e^{-j\theta} = |z||z^*| = |z|^2$.

   ii. $\frac{1}{z} = \frac{1}{|z|e^{j\theta}} = |z|^{-1}e^{-j\theta} = \frac{1}{|z|}e^{-j\theta}$ ή $\frac{z^*}{|z|^2} = \frac{|z^*|e^{-j\theta}}{|z|^2} = \frac{|z^*|e^{-j\theta}}{|z||z^*|} = \frac{1}{|z|}e^{-j\theta}$.

   (γ') $(z+w)^* = (x+jy+u+vj)^* = (x-jy+u-vj) = z^* + w^*$

   $(zw)^* = (|z|e^{j\angle z}|w|e^{j\angle w})^* = |z|e^{-j\angle z}|w|e^{-j\angle w} = z^*w^*$.

   (δ') $z = |r|e^{j\theta}$, $w = |\rho|e^{j\phi}$

   $zw = |r||\rho|e^{j\theta}e^{j\phi} = |r||\rho|e^{j(\theta+\phi)}$

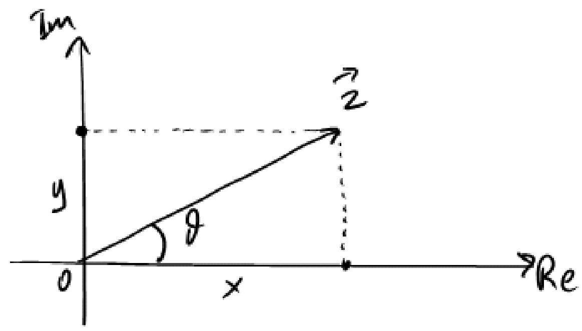
   $zw = (r\cos(\theta) + jr\sin(\theta))(\rho\cos(\phi) + j\rho\sin(\phi) = r\rho(\cos(\theta+\phi) + j\sin(\theta\phi))$

   Προφανώς προτιμάμε την πολική μορφή.

4. **Διανύσματα και μιγαδικοί αριθμοί**

   (α') Ναι, είναι:

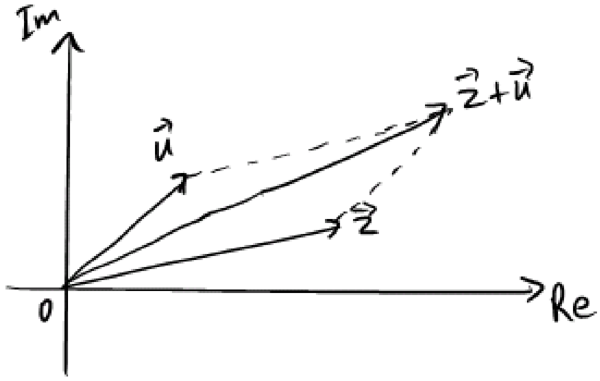
   Έχουμε $|z| = |x|\cos(\theta)$ και $|\cos(\theta)| \leq 1$.



Σχήμα 4: Άσκηση 4α

   Άρα, $|z| \geq |x|$.

   (β') Είναι Έστω $z = 2 + j$ και $u = 1 + 0.5j$. Ισχύει



Σχήμα 5: Άσκηση 4β

5. **Εξίσωση Euler και ορθογωνικότητα ημιτόνων**

(α′)

$$\begin{aligned}
\sin(a)\sin(b) &= \frac{e^{ja}-e^{-ja}}{2j}\frac{e^{jb}-e^{-jb}}{2j} \\
&= \frac{1}{4j^2}(e^{ja}-e^{-ja})(e^{jb}-e^{-jb}) \\
&= -\frac{1}{4}(e^{ja}e^{jb}-e^{ja}e^{-jb}-e^{-ja}e^{jb}+e^{-ja}e^{-jb}) \\
&= -\frac{1}{4}(e^{j(a+b)}-e^{j(a-b)}-e^{j(-a+b)}+e^{-j(a+b)}) \\
&= -\frac{1}{4}(e^{j(a+b)}+e^{-j(a+b)}-e^{j(a-b)}-e^{j(-a+b)}) \\
&= -\frac{1}{4}(2\cos(a+b)-2\cos(a-b)) \\
&= \frac{1}{2}\cos(a-b)-\frac{1}{2}\cos(a+b)
\end{aligned}$$

(β′)
- $x(t+2)=x(t) \Leftrightarrow \cos(\pi(t+2))=\cos(\pi t) \Leftrightarrow \cos(\pi t+2\pi)=\cos(\pi t)$ που ισχύει,γιατί $\cos(2\pi+\theta)=\cos(\theta)$.
- $y(t+2)=y(t) \Leftrightarrow \sin(\pi(t+2))=\sin(\pi t) \Leftrightarrow \sin(\pi t+2\pi)=\sin(\pi t)$ που ισχύει,γιατί $\sin(2\pi+\theta)=\sin(\theta)$.

Άρα $T_0=2$.

$$
\begin{aligned}
\int_{T_0} x(t)y(t)dt &= \int_0^2 \sin(\pi t)\cos(\pi t)dt = \int_0^2 \frac{e^{j\pi t}-e^{-j\pi t}}{2j}\frac{e^{j\pi t}+e^{-j\pi t}}{2}dt \\
&= \frac{1}{4j}\int_0^2 (e^{j\pi t}e^{j\pi t}+e^{j\pi t}e^{-j\pi t}-e^{-j\pi t}e^{j\pi t}-e^{-j\pi t}e^{-j\pi t})dt = \\
&= \frac{1}{4j}\int_0^2 (e^{j2\pi t}-e^{-j2\pi t})dt = \frac{1}{4j}\int_0^2 2j\sin(2\pi t)dt \\
&= \frac{2}{4}\int_0^2 \sin(2\pi t)dt = \frac{1}{2}\int_0^2 \Big(\frac{-1}{2\pi}\cos(2\pi t)\Big)' dt \\
&= -\frac{1}{4\pi}\cos(2\pi t)\Big|_0^2 = -\frac{1}{4\pi}(\cos(4\pi)-\cos(0)) = 0.
\end{aligned}
$$

Άρα τα $x(t), y(t)$ είναι ορθογώνια.

6. **Αθροίσματα ημιτόνων ίδιας συχνότητας**

$$
\begin{aligned}
x(t) &= 2\cos\Big(2\pi 10t+\frac{\pi}{3}\Big)+\sqrt{2}\cos\Big(2\pi 10t-\frac{3\pi}{4}\Big) \\
&= Re\Big\{2e^{j(2\pi 10t+\frac{\pi}{3})}\Big\}+Re\Big\{\sqrt{2}e^{j(2\pi 10t-\frac{3\pi}{4})}\Big\} \\
&= Re\Big\{2e^{j2\pi 10t}e^{j\frac{\pi}{3}}+\sqrt{2}e^{j2\pi 10t}e^{-j\frac{3\pi}{4}}\Big\} \\
&= Re\Big\{(2e^{j\frac{\pi}{3}}+\sqrt{2}e^{-j\frac{3\pi}{4}})e^{j2\pi 10t}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
|A| &= \sqrt{\Big[\Big(2\cos(\frac{\pi}{3})\Big)+\sqrt{2}\cos\Big(-\frac{3\pi}{4}\Big)\Big]^2+\Big[2\sin\Big(\frac{\pi}{3}\Big)+\sqrt{2}\sin\Big(-\frac{3\pi}{4}\Big)\Big]^2} \\
&= \sqrt{\Big(2\frac{1}{2}+\sqrt{2}(-\frac{\sqrt{2}}{2})\Big)^2+\Big(2\frac{\sqrt{3}}{2}+\sqrt{2}\Big(-\frac{\sqrt{2}}{2}\Big)^2\Big)} \\
&= \sqrt{\Big(\frac{2\sqrt{3}}{2}-1\Big)^2} = \sqrt{3}-1
\end{aligned}
$$

$$
\angle A = \tan^{-1}\frac{Im\{z_1\}+Im\{z_2\}}{Re\{z_1\}+Re\{z_2\}} = \tan^{-1}\frac{\sqrt{2}\sin\Big(-\frac{3\pi}{4}\Big)+2\sin\Big(\frac{\pi}{3}\Big)}{0} \quad !!!
$$

Όμως, όταν $\tan^{-1}\frac{y}{x}$, με $y \geq 0$ και $x = 0$, τότε $\tan^{-1} \in [-\frac{\pi}{2}, \frac{\pi}{2})$ με ασύμπτωτες στο $\pm\frac{\pi}{2}$, όταν $x \to 0^{\pm}$.

Άρα

$$
\tan^{-1}\frac{Im\{z_1\}+Im\{z_2\}}{Re\{z_1\}+Re\{z_2\}} = \frac{\pi}{2}.
$$

7. **Σήματα σειρήνας - Chirp signals - MATLAB (bonus 10%)**

(α′)
```
A = 1;
f_c = 1/pi;
t = 0:0.05:40;
s = t.^2/4;
x = A*cos(2*pi*fc*t + s);
```

(b)
```
s = -2*sin(t);
x = A*cos(2*pi*fc*t + s);
```

(c) i. $\frac{d}{dt}\frac{1}{2}\Big(2\pi f_c t + \frac{t^2}{4}\Big) = f_c + \frac{1}{2\pi}\frac{2t}{4} = f_c + \frac{t}{4\pi}.$

ii. $\frac{d}{dt}\frac{1}{2\pi}\Big(2\pi f_c t - 2\sin(t)\Big) = f_c - \frac{1}{\pi}\cos(t).$

```
t = 0:0.05:40;
IF1 = fc + t./(4*pi);
IF2 = fc - cos(t)./pi;
```